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GENERAL THEOREMS OF CLOSURE

S. MANDELBROJT

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By

S. MANDELBROJT

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GENERAL THEOREMS OF CLOSURE, by S. Mandelbrojt, Docteur ès Sciences (Paris), Professor at the Collège de France, Professor of Mathematics at the Rice Institute

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INTRODUCTION

The importance of theorems concerning subspaces of a given vector space (such as L_1) spanned by a function of this space is well known. A now classical result of N. Wiener shows that, in L_1, the closure of the linear combinations of the translations of a function, whose Fourier transform vanishes nowhere, consists of the entire space L_1. In a joint paper, the author and Agmon gave more general theorems on the subspace spanned by a function whose Fourier transform is allowed to have zeros. These propositions, which are quite general, include Wiener's theorem as a very particular case.

On the other hand, the author has recently studied the subspaces spanned by the linear combinations of the successive derivatives (or of a subsequence of the successive derivatives) of a function. More recent results of the author give conditions under which a function can be expressed as the limit, in the norm of the given space, of mixed linear combinations involving the derivatives of the function and the translations of another function (the study has been made in L_1 only). These theorems are not merely combinations of the theorems on the two different types of approximation mentioned above, but require proofs of a different nature. These results and their applications we consider sufficiently interesting to deserve general exposition.

In order to give this *Pamphlet* the character of a monograph we include classical results on Banach spaces and also some considerations on the theory of functions of a complex variable. In particular we give some theorems which, to some extent, generalize a result found by the author and G. MacLane concerning the behavior of a function holomorphic in an infinite strip.

Mr. T. P. G. Liverman has helped me in the editing of this *Pamphlet*. I thank him for his valuable assistance.

Professor G. MacLane has helped me in reading the manuscript and the proof. Some useful remarks were also made by Mr. H. Reiter. I wish to express my gratitude to both of them.

I wish also to thank Miss A. Wheeler for technical help in the execution of this work.

I. PRELIMINARY NOTIONS

1. For the sake of completeness we shall touch upon a few preliminary notions required for the exposition of the main theorems in Chapter III. These notions are classical and we present them briefly, leaving out some of the longer proofs which are to be found in the works quoted in the bibliography to which the numbers in brackets refer.

§1. Normed Vector Spaces

1. A non-void set, E, of elements, called vectors or points (designated by $x, y, z, \cdots$), is called a vector space over the field Φ, of the real numbers or of the complex numbers (designated by $\alpha, \beta, \gamma, \cdots$), if E is an Abelian group under an operation of addition$(+)$ and allows multiplication $(\cdot)$ by the elements of Φ which is associative and doubly distributive over addition in E and Φ and with the property that $1 \cdot x = x$.

Specifically the elements of a vector space E satisfy the following axioms:

G_1: $\quad x + y = y + x$

G_2: $\quad (x + y) + z = x + (y + z)$

G_3: $\quad x + y = x + z$ implies $y = z$

S_1: $\quad \alpha(\beta x) = (\alpha\beta)x$

S_2: $\quad \alpha(x + y) = \alpha x + \alpha y$

S_3: $\quad (\alpha + \beta)x = \alpha x + \beta x$

S_4: $\quad 1 \cdot x = x.$

It is a simple matter to verify that, as a result of these axioms, there is a unique additive identity element θ, called the null vector, satisfying

$$x + \theta = \theta + x = x \qquad \text{for all } x \in E$$

$$0x = \theta \quad \text{and} \quad \alpha\theta = \theta.$$

Also the scalar multiple $(-1)x$ is the unique inverse of x in the group, i.e. $(-1)x + x = \theta$ so one writes $(-1)x = -x$.

2. A subset F of E is called a vector subspace if it is itself a

vector space under the addition and scalar multiplication of E.

The set $x_1, x_2, \cdots, x_n$ of vectors of E is said to *span*, or *generate*, the vector subspace $V \subset E$ if every vector $y \in V$ can be represented as a *linear combination* $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$.

3. The vector space E is *normed* if there is specified a non-negative real-valued function, the *norm*, which for every $x \in E$ has the value $\|x\|$ satisfying the axioms:

N_1: $\|x\| \geqq 0$, and $\|x\| = 0$ if and only if $x = \theta$,

N_2: $\|x + y\| \leqq \|x\| + \|y\|$,

N_3: $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \Phi$ and $x \in E$.

With the norm we can define the *distance* $d(x, y)$ of any two elements $x, y \in E$ as

$$d(x, y) = \|x - y\|,$$

so E is then a metric space and limits have meaning in it: The sequence $\{x_n\}$ has a *limit* $x \in E$ (approaches $x \in E$) if

$$\lim_{n \to \infty} \|x - x_n\| = 0.$$

4. The normed vector space E is termed *complete* if every Cauchy sequence $\{x_n\}$ of vectors in E has a limit point $x \in E$. That is: every sequence $\{x_n\}$ such that

$$\lim_{p,q \to \infty} \|x_p - x_q\| = 0,$$

determines a unique element $x \in E$ with the property

$$\lim_{n \to \infty} \|x - x_n\| = 0.$$

5. A normed vector space which is complete is called a *Banach space*. When Φ is the field of real numbers it is called a real Banach space; it is a complex Banach space when Φ is the field of complex numbers.

A finite or infinite sequence $\{x_n\}$ spans or generates the subspace $V \subset E$ if the set of all linear combinations of elements of $\{x_n\}$ is dense in V, i.e., if every $y \in V$ is a limit point of linear combinations of elements of $\{x_n\}$.

A set of vectors which spans the whole space E is called fundamental.

6. **Examples of Banach spaces.**

a) $R=E^1$, the one dimensional Euclidean space of all real numbers x with $-\infty<x<+\infty$.

The norm is $\|x\|=|x|$.

b) The space (L_∞) of all functions measurable in every finite interval of R and essentially bounded on R.

Here

$$\|x(t)\| = \text{essential upper bound}_{t\in R} |x(t)|,$$

i.e. $\|x(t)\|$ is the positive number m such that $|x(t)|\leqq m$ almost everywhere, and, for every $\epsilon>0$, the set of t for which $|x(t)|\geqq m-\epsilon$ has positive measure.

c) The space $L_p(a, b)$, $p\geqq 1$ (a, b finite or infinite). This is the set of all functions $x(t)$ for which $\int_a^b |x(t)|^p dt<\infty$. The norm is

$$\|x\| = \left(\int_a^b |x(t)|^p dt\right)^{1/p}.$$

In the main body of this paper we shall deal almost exclusively with the space $L_1(-\infty, +\infty)$ or a subspace of this space

§2. Linear Functionals

We restrict ourselves to the bare essentials required to establish theorem R in normed vector spaces. (cf. p. 9)

Henceforth *S and S' designate two normed vector spaces over the same scalar field* Φ (which is either the field C of all complex numbers, or the field R of the reals).

To every $x\in S$ let there correspond a unique $y\in S'$. This correspondence determines a *mapping* or *transformation* $y=T(x)$ of S into S'. When S' is in the space C of complex numbers the transformation is called a *functional*; it is a real functional when $S'\subset R$, the space of real numbers.

$T(x)$ is a *linear transformation* when it is simultaneously *Additive*; i.e. $T(x+x')=T(x)+T(x')$, and *Homogeneous*; i.e. $T(\alpha x)=\alpha T(x)$.

The transformation $T(x)$ is continuous at the point y if and only

if $\lim_{n\to\infty} \|x_n - y\| = 0$ *implies that* $\lim_{n\to\infty} \|T(x_n) - T(y)\| = 0$. It is an immediate consequence of the vector space structure of S that the continuity of $T(x)$ at one point implies its continuity at every point of S. Also *a continuous additive transformation of* S *into* S' *is obviously homogeneous-*R; *i.e.,* $T(\alpha x) = \alpha T(x)$ *for* $\alpha \in R$.

When there exists a fixed positive number M such that $\|T(x)\| \leqq M\|x\|$ for every $x \in S$, the *transformation* $T(x)$ *is bounded.*

LEMMA 1. *An additive transformation* $T(x)$ *of one normed vector space* S *into another,* S', *is continuous if and only if it is bounded.*

Proof: If $T(x)$ is bounded then

$$\|T(x_n) - T(y)\| \leqq M\|x_n - y\|,$$

and hence

$$\lim_{n\to\infty} \|x_n - y\| = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|T(x_n) - T(y)\| = 0.$$

Conversely if $T(x)$ is not bounded there must exist a sequence $\{x_n\}$ for which $\|T(x_n)\| \geqq n\|x_n\|$, and hence

$$\frac{\|T(x_n)\|}{n\|x_n\|} > 1. \tag{1}$$

Taking $y_n = x_n/(n\|x_n\|)$, we have $\lim_{n\to\infty} \|y_n\| = \lim\ (1/n) = 0$, so that if $T(x)$ were continuous (and therefore homogeneous by a previous remark), we would have, in contradiction to (1),

$$\lim_{n\to\infty} \|T(y_n)\| = \lim_{n\to\infty} \frac{\|T(x_n)\|}{n\|x_n\|} = 0.$$

This property leads us to define the *bound*, or *norm* $\|T\|_V$ *of a linear transformation* $T(x)$ on the vector subspace $V \subset S$ as

$$\|T\|_V = \text{l.u.b.}_{x\in V} \frac{\|T(x)\|}{\|x\|}.$$

The following theorem of Banach is fundamental, its proof, too long to be reproduced here, will be found in [3] and [7].

THEOREM 1. *Given a real vector space* S *and*

1) *a real functional* $p(x)$ *defined on* S *which, for all* x *and* $y \in S$, *satisfies* $p(x+y) \leqq p(x)+p(y)$ *and* $p(\alpha x) = \alpha p(x)$ *for all* $\alpha \geqq 0$,

2) *a real linear functional* $f(x)$ *defined on the vector subspace* $V \subset S$ *and which, for all* $x \in V$, *satisfies the relation* $f(x) \leqq p(x)$.

Under these conditions there exists a linear functional $F(x)$, *defined on the entire space* S, *which extends* $f(x)$ *to* S *in the sense that*

$$F(x) = f(x) \qquad \text{for all } x \in V$$

$$F(x) \leqq p(x) \qquad \text{for all } x \in S.$$

From this follows the important

Hahn-Banach Theorem. $f(x)$ *being a continuous real linear functional defined on the vector subspace* V *of the real normed vector space* S, *there exists a real linear continuous functional* $F(x)$, *defined on* S, *which extends* $f(x)$ *to* S *and has the same norm. In other words*

$$F(x) = f(x) \qquad \text{for all } x \in V$$

and

$$\|F\|_S = \|f\|_V.$$

Proof: Choose $p(x) = \|f\|_V \cdot \|x\|$ in Theorem 1. Then there exists a linear functional $F(x)$ such that

$$-\|f\|_V \cdot \|x\| = -p(-x) \leqq F(x) \leqq p(x) = \|f\|_V \cdot \|x\|.$$

Thus $F(x)$ is bounded and (Lemma 1) continuous.

The corresponding result for complex linear functionals is stated in the

Bohnenblust-Sobczyk Theorem. $f(x)$ *being a continuous complex linear functional defined on the vector subspace* V *of the complex normed vector space* S, *there exists a continuous complex linear functional* $F(x)$ *defined on* S *and such that* $f(x) = F(x)$ *on* V *and* $\|F\|_S = \|f\|_V$.

Proof: [7]. The bounded linear complex functional $f(x)$ defined on V can be written as $f(x) = f_1(x) + if_2(x)$. It can easily be checked that $f_1(x)$ and $f_2(x)$ are both bounded linear real-valued

functionals on V and that $\|f_1\|_V \leqq \|f\|_V$, and $\|f_2\|_V \leqq \|f\|_V$. Also f_1 and f_2 being real-valued, we have, for all $x \in V$,

$$f(ix) = f_1(ix) + if_2(ix) = if(x) = if_1(x) - f_2(x),$$

and hence

$$f_1(ix) = -f_2(x).$$

In accordance with the previous theorem there exists a real linear bounded functional $F_1(x)$ on S which coincides with $f_1(x)$ throughout V and for which $\|F_1\|_S = \|f_1\|_V$. Consider the complex linear bounded functional

$$F(x) = F_1(x) - iF_1(ix),$$

which coincides with $f(x)$ on V. Setting $F(x) = \rho e^{i\vartheta}$ gives

$$|F(x)| = e^{-i\vartheta}F(x) = F(e^{-i\vartheta}x) = F_1(e^{-i\vartheta}x) \leqq \|f_1\|_V \|x\| \leqq \|f\|_V \|x\|.$$

§3. Linear Manifolds

Vector subspaces or *linear manifolds*, as they are also called, form the principal subject matter of this *Pamphlet*. E being a vector subspace of S, the symbol $\overline{E}$ is used to designate its *closure*, i.e. the linear manifold consisting of E and all its limit points. Obviously $\overline{E}$ is a normed vector space in its own right. A manifold is closed if it contains all its limit points.

As usual, *linear combination of elements* will always mean a *linear combination of a finite number of elements*. The closed linear manifold $\overline{E}$ is spanned or generated by the sequence $\{e_\nu\}$ if every $x \in \overline{E}$ is the limit of a sequence of linear combinations of elements of $\{e_\nu\}$. In other words $\{e_\nu\}$ *spans* $\overline{E}$ *when the linear combinations of elements of* $\{e_\nu\}$ *are dense in* $\overline{E}$. *This means that to every arbitrary* $\epsilon > 0$ *corresponds a finite integer* $n = n(\epsilon)$ *and a set of scalars* $\alpha_{\nu,n}$ *such that*

$$\text{(C)} \qquad \left\| x - \sum_{\nu=1}^{n} \alpha_{\nu,n} e_\nu \right\| < \epsilon.$$

The linear continuous functional f is called *orthogonal* to $x_0 \in S$ if $f(x_0) = 0$; *if* $f(x) = 0$ *for all* $x \in V \subset S$ *we say that* f *is orthogonal to the set* V.

The concept of orthogonality between linear continuous functionals and vectors provides a basic criterion for the determination of dense subsets in normed vector spaces. This criterion is embodied in Theorem R for whose proof we require the following lemma of Banach:

LEMMA 2. *If E is a closed proper vector subspace of the normed vector space S there exists a non-null continuous linear functional on S which is orthogonal to E.*

Specifically, if there is a point $x_0 \in S$ at the positive distance d from the linear manifold $E \subset S$ (i.e. g.l.b.$_{x \in E} \|x - x_0\| = d > 0$) *there exists a bounded linear functional $F(x)$ on S with the properties*

(a) $$F(x_0) = 1,$$

(b) $$F(x) = 0, \qquad \text{for } x \in E, \text{ and}$$

(c) $$\|F\|_S = 1/d.$$

Indeed, consider the set G of all $y \in S$ of the form

(2) $$y = x + \alpha x_0, \qquad \alpha \in \Phi,\ x \in E.$$

G is a linear manifold and $d > 0$ implies that each $y \in G$ has a unique representation in the form (2) (otherwise we would have $x_1 - x_2 + (\alpha_1 - \alpha_2)x_0 = \theta$, i.e. $x_0 - (x_2 - x_1)/(\alpha_1 - \alpha_2) = \theta$, while the norm of this vector is $\geqq d$). Now $f(y) = \alpha$ defines an additive functional on G, and from

$$\|y\| = \|x + \alpha x_0\| = |\alpha| \, \|x/\alpha + x_0\| \geqq |\alpha| \, d,$$

it follows that

$$|f(x)| = |\alpha| \leqq \|x\|/d,$$

and so

(3) $$\|f\|_G \leqq 1/d.$$

Next consider $\{x_n\} \subset E$ with $\lim_{n \to \infty} \|x_n - x_0\| = d$, then

$$|f(x_n - x_0)| = 1 \leqq \|x_n - x_0\| \, \|f\|_G,$$

and

(4) $$\|f\|_G \geqq 1/d.$$

This proves that $\|f\|_G = 1/d$. The scalar field Φ being either R or C we apply the Hahn-Banach or the Bohnenblust-Sobczyk theorem to assert that $f(x)$ can be extended by a linear bounded functional $F(x)$ on S with $F(y) = f(y)$ on G and $\|F\|_S = \|f\|_G = 1/d$. Since $f(x) = 0$ when $x \in E$, and $f(x_0) = 1$, the proof is complete.

THEOREM R. *A necessary and sufficient condition that a sequence $\{b_\nu\}$ of vectors span the closed linear manifold E of the normed vector space S is that every continuous linear functional $f(x)$ on S, which is orthogonal to every b_ν, be orthogonal to the entire manifold E.*

This can also be stated as follows:

Given an arbitrary set B in the normed vector space S and an element $y \in S$, a necessary and sufficient condition for y to be expressible as a limit of linear combinations, $\{p_\nu\}$, of elements of B is that every continuous linear functional $f(x)$ on S which is orthogonal to B also be orthogonal to y.

Proof: The necessity of the condition is obvious, for if $f(x) = 0$ when $x \in B$ then $f(p_n) = 0$ as well and so $f(\lim p_n) = f(y) = 0$.

That the condition is sufficient follows from Lemma 2, for if y were not a limit of linear combinations $p_n \in B$ there would exist a continuous linear functional $F(x)$ for which $F(x) = 0$ when $x \in B$ while $F(y) = 1$.

Remarks. 1. The fundamental theorem of Chapter III, which constitutes the core of this paper, hinges very essentially on Theorem R.

2. A more direct but computational proof of a theorem of F. Riesz, to which Theorem R is a corollary, is to be found in [4]. We have chosen to follow the present mode of exposition, in spite of the length of the preliminaries required, because it points up the essential concept of orthogonality and can be extended to vector spaces of a more general nature than the ones we have discussed.

§4. Representation of Linear Functionals

In [3] will be found the proofs of Theorems 5 and 6 which are too long to be reproduced here.

Notation. $L_p(a, b)$ designates the Banach space of all functions $x(t)$ such that $\int_a^b |x(t)|^p dt < \infty$, $1 \leqq p < \infty$; a and b may each be finite or infinite.

$L_\infty(a, b)$ designates the Banach space of all functions $x(t)$ essentially bounded and measurable on (a, b) (measurable in every finite interval when a and b are infinite) cf. examples, §1, this chapter. The symbols L_p and L_∞ without indications of an interval will be taken to stand for $L_p(-\infty, \infty)$ and $L_\infty(-\infty, \infty)$ respectively.

THEOREM 5. *Every continuous linear functional $f(x)$ on the space $L_1(a, b)$ can be represented by an expression of the form*

$$f(x) = \int_a^b x(t)\alpha(t)dt$$

where $\alpha(t) \in L_\infty(a, b)$ and $\|f\| =$ ess. up. bd. $|\alpha(t)|$.

THEOREM 6. *Every linear continuous functional, $f(x)$, on the space $L_p(a, b)$, where $1 < p < \infty$, can be represented in the form*

$$f(x) = \int_a^b x(t)\alpha(t)dt,$$

where $\alpha(t) \in L_q(a, b)$, $p^{-1} + q^{-1} = 1$, and except perhaps when L_p is real and f is complex, the norm $\|f\| = (\int_a^b |\alpha(t)|^q dt)^{1/q}$.

In [3] Theorems 5 and 6 are proved for f real and a, b finite, but the passage to $L_p(-\infty, \infty)$ $(p \geqq 1)$, f real or complex, is simple. Suppose first that f is real and defined on L_p real, and define f_n on $L_p(-n, n)$, $(n > 0)$ by $f_n(x) = f(x_n)$, where x_n is the extension of $x \in (-n, n)$ on $(-\infty, \infty)$, i.e., $x_n(t) = x(t)$ for $t \in I_n \equiv (-n, n)$ and $x_n(t) = 0$ for $t \notin I_n$. From Theorems 5 and 6 on $L_p(-n, n)$ it follows that $f_n(x) = \int_{-n}^{n} x(t)\alpha_n(t)dt$ with $\alpha_n \in L_q(-n, n)$, $\|\alpha_n\| = \|f_n\| \leqq \|f\|$. It follows also from $f_n(x) = f_{n+1}(x')$ where x' is the extension of x on I_{n+1}, that on I_n, $\alpha_n(t) = \alpha_{n+1}(t)$ almost everywhere. From the continuity of f it then follows that, on denoting by $x^{(n)}$ the restriction of $x \in L_p$ on I_n, we have $f(x) = \lim_{n\to\infty} f_n(x^{(n)}) = \int_{-\infty}^{\infty} x(t)\alpha(t)dt$ with $\alpha(t) = \alpha_n(t)$ for $t \in I_n$, and $\|\alpha\| \leqq \|f\|$. Since, by Hölder's inequality $|f| \leqq \|x\|\|\alpha\|$, we see that $\|f\| = \|\alpha\|$.

It is now immediately seen that $f(x) = \int x(t)\alpha(t)dt$, with $\alpha \in L_q$, for f complex, L_p real or complex.

If $f \equiv 0$ then obviously $\alpha \equiv 0$ (almost everywhere). If $p = 1$ and $f \not\equiv 0$, then to each m' with $0 < m' < m = \|\alpha\|$ there corresponds a square C with sides parallel to the axes and center at $me^{i\theta}$, where θ is a certain real number, such that C lies outside the circle of radius m' about the origin. There is also a set A of positive measure a such that for $t \in A$, $\alpha(t) \in C$. If $x(t) = 1$ for $t \in A$ and $x(t) = 0$ for $t \notin A$, then $|f(x)| = |\int_A \alpha(t)dt| \geqq am'$ with $\|x\| = a$. Thus $\|f\| \geqq m'$, which proves again that $\|f\| = \|\alpha\|$.

If $p > 1$ and if f is either real and defined on real L_p (α is then real) or is (real or complex) defined on complex L_p then on choosing $x(t)$ such that $x(t)\alpha(t) = |\alpha(t)|^q$, we have $|x(t)|^p = |\alpha(t)|^{p(q-1)} = |\alpha(t)|^q$ and

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} x(t)\alpha(t)dt = \int_{-\infty}^{\infty} |\alpha(t)|^q dt \\ &= \left(\int_{-\infty}^{\infty} |\alpha(t)|^q dt\right)^{1/q} \left(\int_{-\infty}^{\infty} |\alpha(t)|^q dt\right)^{1/p} \\ &= \left(\int_{-\infty}^{\infty} |\alpha(t)|^q dt\right)^{1/q} \left(\int_{-\infty}^{\infty} |x(t)|^p dt\right)^{1/p} \\ &= \|x\| \cdot \|\alpha\|, \end{aligned}$$

and again $\|f\| = \|\alpha\|$.

If $p > 1$ and if f, complex, is defined only on L_p real, then it is not true in general that $\|f\| = \|\alpha\|$. If, for instance, $p = 2$, $\alpha(t) = 1$ for $0 < t \leqq 1$, $\alpha(t) = i$ for $1 < t \leqq 2$, $\alpha(t) = 0$ for $t \leqq 0$ and $t > 2$, then on setting $f(x) = \int_{-\infty}^{\infty} \alpha(t)x(t)dt$ we have

$$f(x) = \int_0^1 x(t)dt + i\int_1^2 x(t)dt.$$

That is to say, for x real:

$$\begin{aligned} |f(x)|^2 &= \left(\int_0^1 x(t)dt\right)^2 + \left(\int_1^2 x(t)dt\right)^2 \\ &\leqq \int_0^1 x^2(t)dt + \int_1^2 x^2(t)dt \leqq \|x\|^2, \end{aligned}$$

and $\|\alpha\| = 2^{1/2}$. We have therefore

$$|f| \leqq 2^{-1/2}\|x\| \cdot \|\alpha\|, \quad \text{and} \quad \|f\| \leqq 2^{-1/2}\|\alpha\|.$$

§5. Fourier Transforms

With every function $f(x) \in L_1$, we can associate its *Fourier transform*, $S(f)$, defined by

$$S(f) = \phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixu} f(x)\,dx. \tag{5}$$

$\phi(u)$ is a bounded continuous function of the real variable u which approaches zero as $u \to \pm\infty$. *The Fourier transforms of the functions of L_1 form the class A.* The correspondence between A and L_1 is one to one: a function of L_1 whose transform is zero vanishes almost everywhere.

The Fourier transform can be inverted: to $\phi(u) \in A$ corresponds

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixu} \phi(u)\,du \tag{6}$$

such that $\phi(u) = S(f)$. The integral in (6) must be interpreted in some generalized sense (see, for instance, p. 14) when $\phi \notin L_1$.

Given two functions, $f_1(x)$ and $f_2(x)$, both $\in L_1$, we adopt the symbol $f_1 * f_2$ to represent the function

$$F(y) = \int_{-\infty}^{\infty} f_1(y - x) f_2(x)\,dx = \int_{-\infty}^{\infty} f_2(y - x) f_1(x)\,dx, \tag{7}$$

which is also a function of L_1. $F(y) = f_1 * f_2 = f_2 * f_1$ *is known as the "Faltung" or convolution of $f_1(x)$ and $f_2(x)$.* Convolution is obviously commutative and associative. A highly useful property of this operation is its conversion by the Fourier transformation into an ordinary product of functions $\in A$. This we proceed to verify by computing $S(f_1 * f_2)$:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) e^{-iyu}\,dy \\
&\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyu}\,dy \int_{-\infty}^{\infty} f_1(y - x) f_2(x)\,dx \\
&\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} e^{-i(y-x)u} f_1(y - x) e^{-ixu} f_2(x)\,dx \\
&\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itu} f_1(t)\,dt \int_{-\infty}^{\infty} f_2(x) e^{-ixu}\,dx.
\end{aligned}$$

All the integrals here are in L_1 and the inversion of the order of integration is justified by Fubini's theorem. We have proved the important result:

$$S(f_1 * f_2) = \sqrt{2\pi}\, S(f_1) \cdot S(f_2), \qquad (f_1, f_2 \in L_1). \tag{8}$$

The Fourier transformation, however, is not limited to the space L_1 in its applicability; it has been generalized to the spaces L_p and Carleman, among others, discussed its application to functions $f(x)$ for which

$$\int_0^x |f(t)|\, dt = O(|x|^k) \quad \text{as} \quad |x| \to \infty, \qquad (k > 0, \text{ constant}).$$

THEOREM 7. *Given $N(u) \in A$ and $D(u) \in A$ such that:*

a) *$D(u)$ does not vanish in an open interval containing the closed interval $[a, b]$,*

b) *$N(u) \equiv 0$ everywhere outside $[a, b]$;*

then

$$R(u) = N(u)/D(u) \in A,$$
$$R(u) = 0 \quad \text{for} \quad u \notin [a, b].$$

Actually this is an immediate corollary of a more sophisticated proposition, viz., consider $g(u) \in A$ which maps the finite interval $a \leqq u \leqq b$ onto the curve $\mathcal{L}$. If $\psi(z)$ is a regular analytic function on $\mathcal{L}$ there exists a function $R(u) \in A$ such that $R(u) = \psi[g(u)]$ in $[a, b]$.

The proof of these results appears in [4].

Next we shall view Carleman's application of Theorem 7 to the study of the "divisors" of zero in a convolution. We agree to adopt the symbol $\Omega(f)$ to represent the closed (when $f \in L_1$) set of points on the real axis R where $S(f) = \phi(u)$ vanishes.

The Fourier integral $S(f)$ of $f \in L_1$ can be split into the difference of two integrals,

$$S(f) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-ixu} f(x)\, dx - \left(- \int_0^{\infty} e^{-ixu} f(x)\, dx \right) \right]. \tag{9}$$

Under quite general conditions, even when $f \notin L_1$ (e.g. when $\int_0^x |f(t)|\, dt = O(|x|^k)$), a meaning can be ascribed to this differ-

ence although the integrals involved may not converge. Thus Carleman considers the functions:

$$F^+(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-ix\zeta} f(x)\,dx, \qquad (\zeta = \xi + i\eta) \tag{10a}$$

$$F^-(\zeta) = -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ix\zeta} f(x)\,dx, \tag{10b}$$

which, providing $\int_0^x |f(t)|\,dt = O(|x|^k)$, are holomorphic in the upper half-plane $\Im(\zeta) > 0$ and in the lower half-plane $\Im(\zeta) < 0$ respectively. When $f \in L_1$ the difference

$$F^+(\xi + i\eta) - F^-(\xi - i\eta) \qquad (\eta > 0) \tag{A}$$

approaches $\phi(\xi) = S(f)$ as $\eta \to 0$. This limit, in an appropriately generalized sense, may also exist when $f \notin L_1$. This amounts to performing an Abelian summation on the difference in (9). It is plain that when, across an interval (α, β), F^+ is the analytic continuation of F^-, the difference (A) tends to zero in (α, β) when $\eta \to 0$. Furthermore this limit is approached uniformly relative to ξ in every closed interval inside (α, β). The converse is also true.

Now, if f_1 and $f_2 \in L_1$ and $f_1 * f_2 = 0$ it follows from (8) that $S(f_1)S(f_2) = 0$; hence if on (α, β), $S(f_1) \neq 0$ we must have $S(f_2) = 0$ in this same interval. Conversely if $S(f_1) = 0$ on (α, β) then $S(f_1 * f_2) = 0$ there also.

From these remarks it is *intuitively* natural to expect the following propositions to hold: if, without f belonging to L_1 (e.g. $f \in L_\infty$), we assume $f_1 \in L_1$ and $S(f_1) \neq 0$ on (α, β) and if $f_1 * f = 0$, then the function F^+ in (10a) is the analytic continuation of F^- across (α, β). This is the meaning of Carleman's lemma given below. Similarly, if f_2 does not belong to L_1 (e.g. $f_2 \in L_\infty$) and $f_1 \in L_1$, while $S(f_1) = 0$ in (α, β), then the function F^+, defined from $f = f_1 * f_2$ as in (10a), is the analytic continuation of F^- across (α, β) and vice-versa. This is the import of the lemma of Agmon and Mandelbrojt [14] which follows Carleman's lemma. In these two intuitive propositions (made rigorous below) the limit of (A), where it exists, plays the rôle of $S(f)$.

Carleman's Lemma. *If $f(x) \in L_\infty$ is a solution of the equation*

(11) $$K * f = 0$$

where $K(x) \in L_1$, then the function $F^+(\zeta)$ is holomorphic in the half-plane $\Im(\zeta) > 0$ and the function $F^-(\zeta)$ is holomorphic in the half-plane $\Im(\zeta) < 0$. Furthermore these two functions are analytic continuations of each other across every closed interval in $R - \Omega(K)$.

Proof: $R - \Omega(K)$ is an open subset of R. In any open interval I of $R - \Omega(K)$ take the closed interval $[a, c]$; throughout $[a, c]$ the transform

(12) $$g(u) = S(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixu} K(x) dx$$

never vanishes.

Define the function $\gamma(u) \in L_1$ as follows:

(13) $$\gamma(u) = \begin{cases} (b - u)(u - a) & \text{for } a \leqq u \leqq b,\ b \in [a, c], \\ 0 & \text{for } u \notin [a, b]. \end{cases}$$

Because of Theorem 7 there exists a function $\psi(x, b) \in L_1$, such that

$$\psi(x, b) = \frac{1}{\sqrt{2\pi}} \int_a^b \frac{\gamma(u)}{g(u)} e^{-ixu} du.$$

Multiplying equation (11) by this function and integrating yields

(14) $$\int_{-\infty}^{\infty} \psi(x, b) dx \int_{-\infty}^{\infty} K(x - t) f(t) dt = 0,$$

where, the integrand being absolutely integrable, we may invert the order of integration. Thus (14) becomes

$$\int_{-\infty}^{\infty} f(t) dt \int_{-\infty}^{\infty} \psi(x, b) K(x - t) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) dt \int_{-\infty}^{\infty} K(x - t) dx \int_a^b \frac{(b - u)(u - a)}{g(u)} e^{-iux} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) dt \int_a^b \frac{(u - a)(b - u)}{g(u)} du \int_{-\infty}^{\infty} K(x) e^{-iu(x+t)} dx = 0,$$

and this, by (12), simplifies to

$$\int_{-\infty}^{\infty} f(t)dt \int_a^b (u-a)(b-u)e^{-iut}du = 0. \tag{15}$$

The holomorphic functions

$$F^+(\zeta) = \int_{-\infty}^{0} e^{-i\zeta t} f(t)dt, \qquad \mathfrak{I}(\zeta) > 0, \text{ and} \tag{10a}$$

$$F^-(\zeta) = -\int_0^{\infty} e^{-i\zeta t} f(t)dt, \qquad \mathfrak{I}(\zeta) < 0 \tag{10b}$$

are majorized in their respective half-planes of regularity by (ess sup $|\phi| = M$)

$$M\int_{-\infty}^{0} \exp\,[t\mathfrak{I}(\zeta)]dt = \frac{M}{\mathfrak{I}(\zeta)}, \text{ and } M\int_0^{\infty} \exp\,[t\mathfrak{I}(\zeta)]dt = \frac{M}{|\,\mathfrak{I}(\zeta)\,|}.$$

The associated functions

$$\begin{aligned} A^+(\zeta) &= \int_a^{\zeta} (\zeta - z)(z-a)F^+(z)dz, \quad \text{and} \\ A^-(\zeta) &= \int_a^{\zeta} (\zeta - z)(z-a)F^-(z)dz \end{aligned} \tag{16}$$

are holomorphic in $\mathfrak{I}(\zeta) > 0$ and $\mathfrak{I}(\zeta) < 0$ respectively.

The integrals in (10a) and (10b) are absolutely convergent and the order of integration in (16) may be inverted to give

$$\begin{aligned} A^+(\zeta) &= \int_{-\infty}^{0} f(t)dt \int_a^{\zeta} e^{-izt}(z-a)(\zeta - z)dz, \qquad \mathfrak{I}(\zeta) > 0, \\ A^-(\zeta) &= -\int_0^{\infty} f(t)dt \int_a^{\zeta} e^{-izt}(z-a)(\zeta - z)dz, \quad \mathfrak{I}(\zeta) < 0. \end{aligned} \tag{17}$$

These functions tend to continuous finite limits when ζ approaches a point on the real axis because (when $t \leqq 0$ and $\mathfrak{I}(\zeta) \geqq 0$ or $t \geqq 0$ and $\mathfrak{I}(\zeta) \leqq 0$)

$$\left| \int_a^{\zeta} e^{-izt}(z-a)(\zeta - z)dz \right| \leqq \frac{c(1 + |\,\zeta\,|^3)}{1 + |\,t\,|^2}, \quad (c = \text{constant}), \tag{18}$$

as can be verified by integrating by parts. Indeed, $|f|$ being bounded, (18) proves that $A^+(\zeta)$ and $A^-(\zeta)$ are uniformly continuous in any closed bounded region in the closed half-planes $\mathfrak{I}(\zeta) \geqq 0$ and $\mathfrak{I}(\zeta) \leqq 0$ respectively. In particular, $A^+(b)$ and $A^-(b)$ are finite and equation (15) simply means that, for all b in $[a, c]$,

$$A^+(b) - A^-(b) = 0.$$

This proves that A^+ and A^- are analytic continuations of each other across $[a, c]$. The same property holds for $F^+(\zeta)$ and $F^-(\zeta)$, since it is trivially verifiable that

$$\frac{d^2}{d\zeta^2} A^+(\zeta) = (\zeta - a)F^+(\zeta), \quad \text{and}$$

$$\frac{d^2}{d\zeta^2} A^-(\zeta) = (\zeta - a)F^-(\zeta).$$

What has been done in one open interval I can be repeated in every open interval of $R - \Omega(K)$.

As an application of Carleman's lemma we have another result concerning division:

LEMMA [14]. *Given* $K \in L_1$, $f_0 \in L_\infty$, *let*

$$f(x) = f_0 * K. \tag{19}$$

Then, if $\Omega(K)$ *contains an interval* (α, β), *each of the functions* $F^+(\zeta)$ *and* $F^-(\zeta)$ *related to* $f(x)$ *by* (10a) *and* (10b) *is the analytic continuation of the other across the interior of* (α, β).

Proof: Notice first that $f(x)$ is continuous and $\in L_\infty$, so $F^+(\zeta)$ and $F^-(\zeta)$ exist and are holomorphic in $\mathfrak{I}(\zeta) > 0$ and $\mathfrak{I}(\zeta) < 0$ respectively.

The proof consists in exhibiting a function $\delta_0(x) \in L_1$ such that $f * \delta_0 = 0$, and whose Fourier transform $S(\delta_0)$ does not vanish in (α, β), while it is zero everywhere else. Since $\Omega(K)$ is a closed set containing (α, β), Carleman's lemma yields the result: F^+ can be continued analytically across (α, β) into F^-.

Consider the "triangular" function $\Delta(x)$ defined by:

$$\Delta(x) = \begin{cases} 1 - |x| & \text{for } |x| \leqq 1 \\ 0 & \text{for } |x| \geqq 1; \end{cases}$$

it is the Fourier transform of[1]

$$\delta(x) = \frac{4}{\sqrt{2\pi}} \frac{\sin^2\left(\frac{x}{2}\right)}{x^2}.$$

Therefore, if $\gamma = (\alpha+\beta)/2$ and $l = (\beta-\alpha)/2$, it follows that

$$\Delta_0(x) = \Delta\left(\frac{x-\gamma}{l}\right) = S(\delta_0(x)) = S(le^{i\gamma x}\delta(lx)).$$

Now $\delta_0(x) = O(x^{-2})$, as $|x| \to \infty$, so we may write

$$\frac{1}{\sqrt{2\pi}} f * \delta_0 = \frac{1}{\sqrt{2\pi}} \delta_0 * f_0 * K = \frac{1}{\sqrt{2\pi}} f_0 * (\delta_0 * K).$$

From (8) we get $S(\delta_0 * K) = \Delta_0(u)S(K)$; this, however, is identically zero, implying that $f * \delta_0 = 0$.

II. FUNCTIONS HOLOMORPHIC IN A STRIP

In this chapter we reproduce and extend various results most of which have already appeared elsewhere [12, 17]. The author has made extensive use of them for the solution of many problems (asymptotic representation, quasi-analyticity, moment problem, etc.).

The following theorem is a particular case of a proposition due to Szegö (whose hypotheses do not include continuity on the circumference):

THEOREM 1. *Let* $\phi(\zeta) \not\equiv 0$, *be bounded and holomorphic in the circle* $|\zeta| < 1$ *and continuous in* $|\zeta| \leqq 1$ *except perhaps for the point* $\zeta = 1$. *Then*

$$\int_0^{2\pi} \log |\phi(e^{i\theta})| \, d\theta > -\infty.$$

Indeed, let $|\phi(\zeta)| \leqq M$ and suppose the origin to be a zero of order k; set

$$\psi(\zeta) = \frac{\phi(\zeta)}{M\zeta^k};$$

then $\psi(0) \neq 0$ and $|\psi(\zeta)| \leqq 1$ in $|\zeta| < 1$. Applying Jensen's theorem we find

$$\int_\epsilon^{2\pi-\epsilon} \log |\psi(\rho e^{i\theta})| \, d\theta \geqq \int_0^{2\pi} \log |\psi(\rho e^{i\theta})| \, d\theta \geqq 2\pi \log |\psi(0)|, \quad (\rho < 1, \epsilon < \pi). \tag{1}$$

Now, according to the well-known lemma of Fatou, if $f_n(x) \geqq 0$ (all n) are integrable on (a, b), then

$$\int_a^b \liminf f_n(x) dx \leqq \liminf \int_a^b f_n(x) dx;$$

thus, in (1), we may let $\rho \to 1$ to obtain[2]

$$\int_\epsilon^{2\pi-\epsilon} \log |\psi(e^{i\theta})| \, d\theta \geqq \limsup_{\rho \to 1} \int_\epsilon^{2\pi-\epsilon} \log |\psi(\rho e^{i\theta})| \, d\theta \geqq 2\pi \log |\psi(0)| > -\infty.$$

We need merely let $\epsilon \to 0$ to have the proof complete.

Let $f(w)$, $(w = u + iv)$ be holomorphic and bounded in the half-plane $u > 0$ and continuous in $u \geqq 0$. The circle $|\zeta| < 1$ is mapped onto $u > 0$ by the function $w = (1+\zeta)/(1-\zeta)$ so that $\phi(\zeta) = f(w)$ satisfies the conditions of Theorem 1. Since $\zeta = e^{i\theta}$ corresponds to $u = 0$, $v = \cot \theta/2$, Theorem 1 is translated into

THEOREM 2. *If $f(w) \not\equiv 0$ is bounded and holomorphic in the half-plane $u > 0$ and continuous in $u \geqq 0$, then*[3]

$$\int^{\infty} \frac{\log |f(iv)|}{v^2}\, dv > -\infty \quad \text{and} \quad \int_{-\infty} \frac{\log |f(iv)|}{v^2}\, dv > -\infty.$$

The function $w = e^z$ maps the strip $|y| < \pi/2$ of the $z = x + iy$ plane onto the half-plane $u > 0$, consequently the above theorem can be adapted to such strips:

THEOREM 3. *When $F(z) \not\equiv 0$ is bounded and holomorphic in the strip $|y| < \pi/2$ and continuous in $|y| \leqq \pi/2$, then*

$$(2) \qquad \begin{aligned} &\int^{\infty} \log \left| F\left(x + i\frac{\pi}{2}\right) \right| e^{-x} dx > -\infty, \\ &\int^{\infty} \log \left| F\left(x - i\frac{\pi}{2}\right) \right| e^{-x} dx > -\infty. \end{aligned}$$

Taking $f(w) = F(\log w)$, this is seen to be equivalent to Theorem 2.

Our purpose being the study of functions holomorphic in strips of non-constant width we shall state this last proposition in a weaker form which will be generalized to strips of a fairly irregular nature.

THEOREM 4. *If $F(z)$ satisfies the hypotheses of Theorem 3 and if $N(x)$ is a non-decreasing function such that*

$$\log |F(x + iy)| \leqq -N(x),$$

then

$$\int^{\infty} N(x) e^{-x} dx < \infty.$$

Prior to proving an analogous proposition for strips of variable width we shall derive some important results on conformal mappings.

Consider two functions $G_1(\sigma)$ and $G_2(\sigma)$, positive and continuous for $\sigma > \sigma_0 \geqq -\infty$ and such that $\lim_{\sigma\to\infty} G_1(\sigma) = G_1 < \infty$, $\lim_{\sigma\to\infty} G_2(\sigma) = G_2 < \infty$. Let Δ_s be the region of the $s = \sigma + it$ plane defined by:

$$\Delta_s: \quad \sigma > \sigma_0, \quad -G_2(\sigma) < t < G_1(\sigma).$$

$z = z(s) = x(s) + iy(s)$ designates a function which maps Δ_s conformally onto the domain D_z: $|y| < \pi/2$ in the z plane, and which is such that $\lim_{\sigma\to\infty} x(s) = \infty$. $s(z) = \sigma(z) + it(z)$ is the inverse function of $z(s)$.

Set

$$\underline{x}(\sigma) = \inf_{-G_2(\sigma)<t<G_1(\sigma)} x(\sigma + it), \qquad \bar{x}(\sigma) = \sup_{-G_2(\sigma)<t<G_1(\sigma)} x(\sigma + it),$$

$$\underline{\sigma}(x) = \inf_{|y|<\pi/2} \sigma(x + iy), \qquad \bar{\sigma}(x) = \sup_{|y|<\pi/2} \sigma(x + iy),$$

and

$$G(\sigma) = G_1(\sigma) + G_2(\sigma).$$

We next prove two special cases of two now classical theorems of Ahlfors [1]. The proofs are adaptations of those used by Mlle. J. Ferrand and L. Dufresnoy for the general theorems [5, 5a].

THEOREM 5. *If σ_1 and σ_2, $(\sigma_0 < \sigma_1 < \sigma_2)$ are such that*

$$\int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} > 2, \tag{3}$$

then

$$\underline{x}(\sigma_2) - \bar{x}(\sigma_1) \geqq \pi \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} - 4\pi.$$

Let $\omega(\sigma) = \bar{x}(\sigma) - \underline{x}(\sigma)$; the image of the segment $-G_2(\sigma) < t < G_1(\sigma)$ of the vertical line with the abscissa σ has a length at least equal to $(\omega^2(\sigma) + \pi^2)^{1/2}$, i.e.

$$(\omega^2(\sigma) + \pi^2)^{1/2} \leqq \int_{-G_2(\sigma)}^{G_1(\sigma)} | z'(\sigma + it) | \, dt.$$

This, by Schwarz's inequality, becomes

$$\omega^2 + \pi^2 \leqq (G_1(\sigma) + G_2(\sigma)) \int_{-G_2(\sigma)}^{G_1(\sigma)} | z'(\sigma + it) |^2 dt$$

$$= G(\sigma) \int_{-G_2(\sigma)}^{G_1(\sigma)} | z'(\sigma + it) |^2 dt,$$

and it follows that:

$$\int_{\sigma_1}^{\sigma_2} \frac{\omega^2(\sigma)}{G(\sigma)} d\sigma + \pi^2 \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} \leqq \int_{\sigma_1}^{\sigma_2} d\sigma \int_{-G_2(\sigma)}^{G_1(\sigma)} | z'(\sigma + it) |^2 dt. \tag{4}$$

The double integral on the right gives the area of the map, under $z = z(s)$ of that part of Δ_s located between the vertical lines with the abscissae σ_1 and σ_2. This area must be $< \pi(\bar{x}(\sigma_2) - \underline{x}(\sigma_1))$, so (4) gives

$$\int_{\sigma_1}^{\sigma_2} \frac{\omega^2(\sigma)}{G(\sigma)} d\sigma + \pi^2 \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} \leqq \pi(\bar{x}(\sigma_2) - \underline{x}(\sigma_1))$$

$$= \pi[\underline{x}(\sigma_2) - \bar{x}(\sigma_1) + \omega(\sigma_2) + \omega(\sigma_1)];$$

hence

$$\underline{x}(\sigma_2) - \bar{x}(\sigma_1) \geqq \pi \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} + \frac{1}{\pi} \int_{\sigma_1}^{\sigma_2} \frac{\omega^2(\sigma)}{G(\sigma)} d\sigma - \omega(\sigma_2) - \omega(\sigma_1). \tag{5}$$

Now (3) implies the existence of σ_1' and σ_2' where $\sigma_1 < \sigma_1' < \sigma_2' < \sigma_2$, such that

$$\int_{\sigma_1}^{\sigma_1'} \frac{d\sigma}{G(\sigma)} = \int_{\sigma_2'}^{\sigma_2} \frac{d\sigma}{G(\sigma)} = 1. \tag{6}$$

Then if we set

$$h_1(\sigma) = \frac{1}{\pi} \int_{\sigma}^{\sigma_1'} \frac{\omega^2(t)}{G(t)} dt - \omega(\sigma),$$

$$h_2(\sigma) = \frac{1}{\pi}\int_{\sigma_2'}^{\sigma} \frac{\omega^2(t)}{G(t)}\,dt - \omega(\sigma),$$

(5) may be altered to

$$\underline{x}(\sigma_2) - \bar{x}(\sigma_1) \geqq \pi \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} + h_1(\sigma_1) + h_2(\sigma_2). \tag{7}$$

Next, we shall prove the existence of $\bar{\sigma}_1$ with $\sigma_1 < \bar{\sigma}_1 < \sigma_1'$, for which $h_1(\bar{\sigma}_1) \geqq -\pi$. Indeed, assuming the contrary would give

$$h_1(\sigma) + \pi + \omega(\sigma) = \pi + \frac{1}{\pi}\int_{\sigma}^{\sigma_1'} \frac{\omega^2(t)}{G(t)}\,dt < \omega(\sigma),$$

so that, letting $\phi(\sigma) = h_1(\sigma) + \pi + \omega(\sigma)$, we would also have

$$-\pi\phi'(\sigma)G(\sigma) = \omega^2(\sigma) > \phi^2(\sigma),$$

or in other terms:

$$\pi \frac{d}{d\sigma}\left(\frac{1}{\phi(\sigma)}\right) = -\pi\frac{\phi'(\sigma)}{\phi^2(\sigma)} > \frac{1}{G(\sigma)},$$

so that

$$\int_{\sigma_1}^{\sigma_1'} \frac{d\sigma}{G(\sigma)} < \pi \int_{\sigma_1}^{\sigma_1'} d\left(\frac{1}{\phi(\sigma)}\right) = \pi\left(\frac{1}{\phi(\sigma_1')} - \frac{1}{\phi(\sigma_1)}\right)$$

$$= 1 - \frac{\pi}{\phi(\sigma_1)} < 1,$$

which contradicts (6). One proves similarly the existence of $\bar{\sigma}_2$, $(\bar{\sigma}_2' < \bar{\sigma}_2 < \sigma_2)$ for which $h_2(\bar{\sigma}_2) \geqq -\pi$.

In (7), substituting $\bar{\sigma}_1$ for σ_1 and $\bar{\sigma}_2$ for σ_2 ,we obtain

$$\underline{x}(\bar{\sigma}_2) - \bar{x}(\bar{\sigma}_1) \geqq \int_{\bar{\sigma}_1}^{\bar{\sigma}_2} \frac{d\sigma}{G(\sigma)} - 2\pi. \tag{8}$$

Now, obviously, $\underline{x}(\sigma_2) - \bar{x}(\sigma_1) > \underline{x}(\bar{\sigma}_2) - \bar{x}(\bar{\sigma}_1)$, and (8) yields

$$\underline{x}(\sigma_2) - \bar{x}(\sigma_1) \geqq \pi \int_{\bar{\sigma}_1}^{\bar{\sigma}_2} \frac{d\sigma}{G(\sigma)} - 2\pi$$

$$= \pi \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} - \pi \left[\int_{\sigma_1}^{\bar{\sigma}_1} \frac{d\sigma}{G(\sigma)} + \int_{\underline{\sigma}_2}^{\sigma_2} \frac{d\sigma}{G(\sigma)} \right] - 2\pi$$

$$\geqq \pi \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} - 4\pi,$$

which we had set out to prove.

As we are dealing with conformal mappings it is plain that $\underline{x}[\bar{\sigma}(x)] \leqq x$ and $\bar{x}[\underline{\sigma}(x)] \geqq x$, i.e. $\underline{x}[\bar{\sigma}(x)] - \bar{x}[\underline{\sigma}(x)] \leqq 0$, so by Theorem 5 we must have

$$\int_{\underline{\sigma}(x)}^{\bar{\sigma}(x)} \frac{d\sigma}{G(\sigma)} \leqq 4. \tag{9}$$

Recalling that $\lim G(\sigma) = G_1 + G_2 = G$, (9) gives

$$\bar{\sigma}(x) - \underline{\sigma}(x) \leqq 4G + o(1), \qquad (x \to \infty),$$

so that for any $|y_1| \leqq \pi/2$ and $|y_2| \leqq \pi/2$:

$$|\sigma(x + iy_1) - \sigma(x + iy_2)| \leqq 4G + \epsilon(x)$$

where $\epsilon(x) \to 0$ as $x \to +\infty$.

The second theorem of Ahlfors, in a restricted form, now follows:

THEOREM 6. *Let $G_1(\sigma)$ and $G_2(\sigma)$ be positive and continuous for $\sigma \geqq \sigma_0 \geqq -\infty$. In addition, let these functions be of bounded variation: $\int_{\sigma_0}^{\infty} |dG_1(\sigma)| < \infty$, and $\int_{\sigma_0}^{\infty} |dG_2(\sigma)| < \infty$, and let $\lim_{\sigma \to +\infty} G_1(\sigma) = G_1 > 0$, $\lim_{\sigma \to +\infty} G_2(\sigma) = G_2 > 0$. Then there exists a constant, C, such that, for sufficiently great σ_1 and $\sigma_2 > \sigma_1 > \sigma_0$,*

$$\bar{x}(\sigma_2) - \underline{x}(\sigma_1) \leqq \pi \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{G(\sigma)} + C. \tag{10}$$

Proof: Consider the variation $V(\sigma)$ of $G(\sigma)$ from σ_0 to $\sigma > \sigma_0$ and the constants $V > V(\sigma)$, $L > G(\sigma)$ (for sufficiently large σ), and $\delta > 0$ such that $G(\sigma) \geqq \delta$.

Let Γ_x designate the image in Δ_s of the vertical segment of D_z whose abscissa is x; let $A(\sigma)$ be the area of that part of Δ_s with abscissae between c and σ, (c, constant, $> \sigma_0$). Let $\alpha(x)$

be such that $A(\alpha(x))$ equals the area of the portion of Δ_s enclosed by Γ_x and the vertical line $\sigma = c$. The greatest lower bound of the lengths of the arcs connecting two points, one on the upper boundary curve $t = G_1(\sigma)$, the other on $t = -G_2(\sigma)$, and which intersect the vertical line with abscissa σ will be designated by $m(\sigma)$. The vertical line $\sigma = \alpha(x)$ intersects Γ_x, consequently $m(\alpha(x))$ cannot be greater than the length of Γ_x. This we write as

$$m(\alpha(x)) \leqq \int_{-\pi/2}^{\pi/2} | s'(x + iy) | \, dy,$$

and, by Schwarz's inequality,

$$m^2(\alpha(x)) \leqq \pi \int_{-\pi/2}^{\pi/2} | s'(x + iy) |^2 dy. \tag{11}$$

$A(\alpha(x))$ is the area of a portion of Δ_s which is the map of $D(x) \subset D_z$. $D(x)$ is clearly the part of D_z bounded by the map of the line $\sigma = c$ in Δ_s and by the vertical line with abscissa x; therefore we have

$$A(\alpha(t)) = \iint_{D(t)} | s'(z) |^2 dx dy,$$

and

$$\frac{d}{dx} A(\alpha(x)) = \int_{-\pi/2}^{\pi/2} | s'(x + iy) |^2 dy,$$

which, with (11), yields

$$m^2(\alpha(x)) \leqq \pi \frac{d}{dx} A(\alpha(x)).$$

Since $A'(\sigma) = G(\sigma)$, $\alpha(x)$ is differentiable; we then find that

$$m^2(\alpha(x)) \leqq \pi \frac{dA(\alpha(x))}{d\alpha(x)} \cdot \frac{d\alpha(x)}{dx} = \pi G(\alpha(x)) \alpha'(x).$$

Since $\alpha'(x)$ is positive, the inverse function $x(\alpha)$ of $\alpha(x)$ is well defined and we may write

$$\frac{dx}{d\alpha} \leqq \frac{\pi G(\alpha)}{m^2(\alpha)},$$

which, integrated from x_1 to x_2, $(x_2 > x_1)$, yields

$$x_2 - x_1 \leqq \pi \int_{\alpha(x_1)}^{\alpha(x_2)} \frac{G(\alpha)d\alpha}{m^2(\alpha)}. \tag{12}$$

Recalling the definitions of $\underline{\sigma}(x)$ and $\bar{\sigma}(x)$ it is clear that $\underline{\sigma}(x) \leqq \alpha(x) \leqq \bar{\sigma}(x)$, so that from (9) follow the inequalities:

$$\alpha(\bar{x}(\sigma_2)) - \sigma_2 \leqq \bar{\sigma}(\bar{x}(\sigma_2)) - \underline{\sigma}(\bar{x}(\sigma_2)) < 4L,$$
$$\alpha(\underline{x}(\sigma_1)) - \sigma_1 \geqq \underline{\sigma}(\bar{x}(\sigma_1)) - \bar{\sigma}(\bar{x}(\sigma_1)) > -4L.$$

These, with (12), taking σ_1 sufficiently large, imply

$$\begin{aligned} \bar{x}(\sigma_2) - \underline{x}(\sigma_1) &\leqq \pi \int_{\alpha(\underline{x}(\sigma_1))}^{\alpha(\bar{x}(\sigma_2))} \frac{G(\alpha)}{m^2(\alpha)} d\alpha \leqq \pi \int_{\sigma_1 - 4L}^{\sigma_2 + 4L} \frac{G(\alpha)}{m^2(\alpha)} d\alpha \\ &\leqq \pi \int_{\sigma_1}^{\sigma_2} \frac{G(\alpha)}{m^2(\alpha)} d\alpha + \frac{8\pi L^2}{\delta^2}, \end{aligned} \tag{13}$$

while one may also write:

$$\begin{aligned} \int_{\sigma_1}^{\sigma_2} \frac{G(\alpha)}{m^2(\alpha)} d\alpha &= \int_{\sigma_1}^{\sigma_2} \frac{d\alpha}{G(\alpha)} + \int_{\sigma_1}^{\sigma_2} \frac{G^2(\alpha) - m^2(\alpha)}{G(\alpha)m^2(\alpha)} d\alpha \\ &\leqq \int_{\sigma_1}^{\sigma_2} \frac{d\alpha}{G(\alpha)} + \frac{2L}{\delta^3} \int_{\sigma_1}^{\sigma_2} (G(\alpha) - m(\alpha))d\alpha. \end{aligned} \tag{14}$$

It is obvious, on a diagram, that the sum of the variations of $G_1(\sigma)$ and $G_2(\sigma)$ in $(\alpha - L, \alpha + L)$ dominates $G(\alpha) - m(\alpha)$, i.e. that

$$G(\alpha) - m(\alpha) \leqq V(\alpha + L) - V(\alpha - L).$$

Consequently

$$\begin{aligned} \int_{\sigma_1}^{\sigma_2} (G(\alpha) - m(\alpha))d\alpha &\leqq \int_{\sigma_1}^{\sigma_2} (V(\alpha + L) - V(\alpha - L))d\alpha \\ &\leqq 4LV. \end{aligned} \tag{15}$$

Combining (13), (14) and (15) we have (10). q.e.d.

Now, let $\sigma_1=\sigma_2=\sigma$ in (10); this shows that, under the hypotheses of Theorem 6, there exists a constant, M, such that, for σ sufficiently large, we always have

$$|x(\sigma+it_1)-x(\sigma+it_2)|\leqq M, \tag{16}$$

whatever the values of t_1 and t_2, providing $-G_2(\sigma)<t_1<G_1(\sigma)$ and $-G_2(\sigma)<t_2<G_1(\sigma)$.

Also from (10) follows the existence of two constants M_1 and M_2 such that, σ_1 and σ_2 being sufficiently large, we have

$$|\bar{x}(\sigma_2)-\underline{x}(\sigma_1)|<M_1|\sigma_2-\sigma_1|+M_2. \tag{17}$$

Keeping the notation so far used, we may deduce from Theorems 5 and 6 the following lemma:

LEMMA 1. *Given two positive, continuous functions $G_1(\sigma)$ and $G_2(\sigma)$ of bounded variation for $\sigma\geqq\sigma_0\geqq-\infty$, for which $0<\lim_{\sigma\to\infty}G_1(\sigma)$ and $0<\lim_{\sigma\to\infty}G_2(\sigma)$, and setting*

$$S(\sigma)=\pi\int_c^\sigma\frac{du}{G(u)},\qquad (G(u)=G_1(u)+G_2(u)), \tag{18}$$

where c is constant, $c>\sigma_0$, then

$$\underline{x}(\sigma)+O(1)<S(\sigma)<\bar{x}(\sigma)+O(1) \tag{19}$$

as $\sigma\to+\infty$.

Indeed let $\sigma_1=c$ and $\sigma_2=\sigma$ in Theorems 5 and 6.

Keeping the notation used so far we can now state and prove:

THEOREM 7. *For every ϵ, where $0<\epsilon<\pi/2$, we have uniformly with respect to y when $|y|\leqq\pi/2-\epsilon$,*

$$\lim_{\sigma\to\infty}\frac{d}{dz}s(z)=\frac{G}{\pi},\qquad (G=G_1+G_2=\lim_{\sigma\to\infty}(G_1(\sigma)+G_2(\sigma))).$$

Indeed, the harmonic function $t(x+iy)-Gy/\pi$ is continuous and bounded on the boundary of D_z including the point at infinity ($x\to\infty$) where it vanishes; consequently this function tends to zero uniformly for $|y|\leqq\pi/2$, as $z\in D_z$ approaches the point at infinity ($x\to\infty$). Thus

$$\lim_{x\to\infty} t(x+iy) = \frac{yG}{\pi}, \qquad \left(|y| \leqq \frac{\pi}{2}\right). \tag{20}$$

Consider the rectangle R_0: $|y|<\pi/2$, $|x|<a<\infty$, $(a>0$, constant). For every $h>0$ define:

$$f_h(z) = s(z+h) - s(h).$$

By (20) we obtain, uniformly in R_0,

$$\lim_{h\to\infty} \Im f_h(z) = \lim_{h\to\infty} [t(x+h+iy) - t(h)] = \frac{yG}{\pi}, \tag{21}$$

so the $f_h(z)$, $(h\geqq 0)$ form a normal family in R_0. We also have $f_h(0)=0$. The family is therefore also bounded and (21) gives

$$\lim_{h\to\infty} f_h(z) = \frac{Gz}{\pi}$$

uniformly in every closed set R_1 inside R_0. Therefore, throughout R_1, we also have

$$\lim_{h\to\infty} f_h'(z) = \lim \frac{d}{dz} s(z+h) = \frac{G}{\pi}.$$

The function $S(\sigma)$ being defined by (18), and setting $\sigma_0(x) = \sigma(x+i\pi/2) = \Re s(x+i\pi/2)$, we can prove the following result [12].

Lemma 2. *If $N(\sigma)$ is a non-decreasing function, the relation*

$$\int^{\infty} N(\sigma) e^{-S(\sigma)} d\sigma = \infty \tag{22}$$

implies the relation

$$\int^{\infty} N(\sigma_0(x)) e^{-x} dx = \infty. \tag{23}$$

Indeed, because of (9), we have for sufficiently large x, when $\gamma>4G$

$$\sigma_0(x) - \gamma < \sigma(x) < \sigma_0(x) + \gamma. \tag{24}$$

Now, by Theorem 7, $d\sigma(x)/dx$ is an increasing function for x sufficiently large; we may consequently rewrite (22) as

$$\int^{\infty} N(\sigma(x) - \gamma)e^{-S(\sigma(x)-\gamma)}d\sigma(x) = \infty. \tag{25}$$

From (9) and the relation $\lim G(\sigma) = G > 0$, we find:

$$S(\sigma_0(x) - \gamma) \geqq S(\sigma_0(x)) + p, \qquad (p \text{ constant}), \tag{26}$$

while the inequality on the left-hand side of (24) shows that

$$S(\sigma(x)) \geqq S(\sigma_0(x) - \gamma), \tag{27}$$

and hence, by (26) and (27),

$$S(\sigma(x)) \geqq S(\sigma_0(x)) + p. \tag{28}$$

$N(\sigma)$ is non-decreasing, so the inequality on the right-hand side of (24) combined with (28) implies

$$\int^{\infty} N(\sigma_0(x))e^{-S(\sigma_0(x))}d\sigma(x) = \infty. \tag{29}$$

This, on application of Theorem 7 and (19), yields

$$\int^{\infty} N(\sigma_0(x))e^{-x}dx = \infty,$$

which we wished to prove.

We can now prove the promised generalization of Theorem 4 to functions holomorphic in strips of variable width. It was first given by Mandelbrojt and MacLane for symmetrical strips ($G_1(\sigma) = G_2(\sigma)$); Mlle. Ferrand later pointed out its validity for non-symmetrical strips [6].

THEOREM 8a. *Let $F(s)$ be holomorphic and bounded in the closed region Δ_s: $\sigma \geqq \sigma_0$, $-G_2(\sigma) \leqq t \leqq G_1(\sigma)$, where $G_1(\sigma)$ and $G_2(\sigma)$ are positive continuous functions of bounded variation such that $\lim_{\sigma\to\infty} G_1(\sigma) = G_1 > 0$ and $\lim_{\sigma\to\infty} G_2(\sigma) = G_2 > 0$. Let $N(\sigma)$ be a non-decreasing function such that on Δ_s:*

$$\log |F(s)| < -N(\sigma).$$

Let $c>\sigma_0$ be constant, and set $(G(\sigma)=G_1(\sigma)+G_2(\sigma))$,

$$S(\sigma) = \pi \int_c^\sigma \frac{d\sigma}{G(\sigma)}.$$

Then, if

$$\int^\infty N(\sigma)e^{-S(\sigma)}d\sigma = \infty,$$

the function $F(s)$ vanishes identically.

Indeed, take $\phi(z)=F(s(z))$, $s(z)$ being defined above. Then

$$\begin{aligned} \log \left| \phi(x + i\pi/2) \right| &= \log \left| F(\sigma_0(x) + iG_1(\sigma_0(x))) \right| \\ &\leqq - N(\sigma_0(x)), \end{aligned} \tag{30}$$

where, as before, $\sigma_0(x)=\sigma(x+i\pi/2)$.

$\phi(z)$ is holomorphic and bounded on the closed strip $|y| \leqq \pi/2$, so that by Theorem 3 this function must either satisfy

$$\int^\infty \log \left| \phi(x + i\pi/2) \right| e^{-x}dx > - \infty$$

or else be identically zero. By (30), this last inequality gives

$$\int^\infty N(\sigma_0(x))e^{-x}dx < \infty,$$

and so, by Lemma 2,

$$\int^\infty N(\sigma)e^{-S(\sigma)}d\sigma < \infty.$$

This contradicts the hypotheses; hence $\phi(z)$ and $F(s)$ vanish identically. q.e.d.

It should be noted that the following Theorem, analogous to Theorem 8a, is true.

THEOREM 8b. *The conclusions of Theorem* 8a *remain valid when the hypotheses on $G_1(\sigma)$ and $G_2(\sigma)$ are replaced by the following ones:*

$G_1(\sigma)$ and $G_2(\sigma)$ are bounded above and below by positive constants. They are differentiable and

$$(31)\qquad \int^{\infty} |G_j'(\sigma)|^2 d\sigma < \infty, \quad \int^{\infty} |dG_j'(\sigma)| < \infty, \quad (j = 1, 2).$$

The proof of this result requires the following condensed and particularized version of three theorems of Warschawski [20] (see also the already quoted work of Mlle. Ferrand).

THEOREM 9. *If $G_1(\sigma)$ and $G_2(\sigma)$ satisfy the hypotheses set forth in Theorem* 8b *the conclusions of Theorems* 5 *(the constant 4π being replaced by another constant) and* 6 *remain true and, setting $2g(\sigma) = (G_1(\sigma) - G_2(\sigma))$, we have*

$$(32)\qquad \lim_{\sigma \to \infty} \frac{dz(s)}{ds} G(\sigma) = \pi,$$

uniformly in every domain[4]

$$(33)\qquad \left| \frac{t - g(\sigma)}{G(\sigma)} \right| < \gamma < \tfrac{1}{2}.$$

By means of Theorem 9 we can still prove Lemma 2. It suffices to replace $\sigma(x)$ in (25) by $\sigma_1(x)$ defined by the relation $\mathcal{R}(z(\sigma_1 + ig(\sigma_1))) = x$. Since (32) holds in the domain (33), $d\sigma_1(x)/dx$ is also positive for x sufficiently large. The proof of Lemma 2 can then be carried through with the aid of the inequalities in the conclusions of Theorems 5 and 6 which are valid in the domain Δ_s defined by the new curves $t = G_1(\sigma)$ and $t = -G_2(\sigma)$. Theorem 8b then follows in quite the same way from Lemma 2 as does Theorem 8a.

Theorems 8a and 8b will be required in the sequel for the discussion of properties of closure. We shall apply them to the determination of *associated functions.*

Consider a closed set E on the real axis R in the complex $z = x + iy$ plane. If $E = R$ we designate by Δ_E the upper half-plane: $y > 0$. If E is not the entire line R, Δ_E *represents the part of z-plane left after the removal of the set E.*

Let $M(r)$ be a positive function defined for $r > 0$. $M(r)$ is said to be a function associated with the set E, if every function $\phi(z)$ which is single-valued and holomorphic in Δ_E and satisfies there the inequality

$$|\phi(z)| \leqq \frac{M(|z|)}{|y|}, \tag{34}$$

is identically zero.

Obviously for the upper half-plane may be substituted the lower half-plane in the above definition of Δ_E when $E=R$.

It is also perfectly clear that when $M(r)$ is associated with E_1 it is automatically associated with every set $E_2 \subset E_1$. With the aid of Theorem 8 we shall now derive some criteria for the determination of functions $M(r)$ associated with certain given sets. These criteria are couched in the form of lemmas to be applied in later chapters in proving certain theorems.

LEMMA M_1. *A constant is associated with the void set.*

Indeed $E \subset R$ being the empty set Λ, Δ_E is the whole z-plane. Let $\phi(z)$ be an entire function and satisfy

$$|\phi(z)| \leqq \frac{M}{|y|}, \qquad M \text{ constant.}$$

For $h>0$ and a real, $\phi_{\alpha,\beta}(z)=(z-\alpha)(z-\beta)\phi(z)$ satisfies the inequalities

$$|\phi_{a-h,a+h}(a-h+iy)| \leqq M|2h+iy|,$$
$$|\phi_{a-h,a+h}(a+h+iy)| \leqq M|2h+iy|,$$
$$|\phi_{a-h,a+h}(x \pm iy)| \leqq \frac{M}{|y|}|2h+iy|^2, \qquad (|x-a| \leqq h).$$

Therefore, on the boundary of the rectangle $|y| \leqq 2h$, $a-h \leqq x \leqq a+h$ we have

$$|\phi_{a-h,a+h}(z)| \leqq 4Mh,$$

which shows that, for $|y| \leqq h$, we have

$$|\phi(a+iy)| \leqq 4Mh^{-1}.$$

Then, for all y:

$$|\phi(a+iy)| \leqq 4Mh^{-1}. \tag{35}$$

Since h is arbitrary, we have $\phi(z) \equiv 0$. This proves that M is associated with $E = \Lambda$.

LEMMA M_2. *Every non-increasing function $M(r)$ for which*

$$-\int^{\infty} \log M(e^{\sigma}) e^{-\sigma} d\sigma = \infty \tag{36}$$

is associated with the entire real axis: R.

Here Δ_E is the upper half-plane: $y > 0$. Suppose, therefore, that

$$|\phi(z)| \leqq \frac{M(|z|)}{|y|}, \qquad y > 0,$$

$\phi(z)$ being holomorphic for $y > 0$. When $y \geqq 1$ we have

$$|\phi(z)| \leqq M(|z|).$$

If $z = ie^{s}$, $(|t| < \pi/2)$ then the function $F(s) = \phi(ie^{s})$ is holomorphic in the domain Δ_s defined by

$$|t| \leqq \arccos e^{-\sigma}, \qquad (\sigma \geqq 0, \ |t| < \pi/2),$$

and in this region

$$\log |F(s)| \leqq -\log M(e^{-\sigma}).$$

Now, writing $G_1(\sigma) = G_2(\sigma) = \arccos e^{-\sigma}$, we find

$$G(\sigma) = 2G_1(\sigma), \qquad S(\sigma) = \pi \int_1^{\sigma} \frac{du}{G(u)},$$

$$S(\sigma) - \sigma = \pi \int_1^{\sigma} \left(\frac{1}{G(u)} - \frac{1}{\pi} \right) du - 1 = O(1), \qquad (\sigma \to \infty).$$

So by Theorem 8a, $F(s) \equiv 0$. q.e.d.

Before establishing a sufficient condition ensuring that a given function $M(r)$ is associated with any given closed set $E \subset R$ we define the *h-covering*, E_h *of* E to be the set $E_h = \bigcup_{x \in E} [x-h, x+h]$; thus E_h is the union of all closed intervals of length $2h$ centered on the points of E.

We recall that the *characteristic function* $\phi(x)$ *of the set* $A \subset R$ is defined as

$$\phi(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \in \mathcal{C}A = \text{complement of } A \text{ with respect to } R. \end{cases}$$

LEMMA M_3. *Let E be a non-void closed set on the real axis. Let E_h be an h-covering of E, where h is a positive constant, and let $\phi(x)$ be the characteristic function of $\mathcal{C}E_h$.*

Let $u(x)$ be continuous on R, bounded positively below, and of bounded variation, and let

$$u(x) \leqq \tfrac{1}{2} + \phi(x) + B\phi(x)\phi(-x), \qquad (\text{constant } B > 0).$$

Every non-increasing function $M(r)$ for which

$$(37) \qquad -\int^{\infty} \log M(e^{\sigma}) \exp\left[-\int^{\sigma} \frac{d\tau}{u(e^{\tau}) + u(-e^{\tau})}\right] d\sigma = \infty$$

is associated with the set E.

Proof: Lemma M_2 is a particular case of this, for with $E = R$ it suffices to choose $u(x) = 1/2$. Let us therefore suppose E to be neither the empty set nor the entire line R.

There is no loss of generality in choosing h sufficiently small for $\mathcal{C}E_h$ not to be empty, for otherwise $u(x) \leqq 1/2$, and we would necessarily have

$$-\int^{\infty} \log M(e^{\sigma}) e^{-\sigma} d\sigma = \infty,$$

which would, according to Lemma M_2 associate $M(r)$ with $E_h = R$ and so with $E \subset E_h$. Let D be the set

$$D = \bigcup_{x \in E} \Delta(x, h),$$

where $\Delta(x, h)$ is the open square centered on x with the edge $2h$.

Consider $\Phi(z)$ holomorphic in Δ_E and satisfying there the inequality

$$|\Phi(z)| \leqq \frac{M(|z|)}{|y|}.$$

When $2|a| > h$ and $a \in \mathcal{C}E_{h/2}$ we have

$$|\Phi(a+iy)| \leqq 8h^{-1} M\left(|a| - \frac{h}{2}\right). \tag{38}$$

This follows from (35) where $M(|a| - h/2)$ is substituted for M and $h/2$ for h, for these substitutions in the computations leading to (35) give (38). Consequently we have, on $\mathcal{C}D$ (the complement is taken relative to the entire plane),

$$|\Phi(z)| \leqq LM\left(\frac{|z|}{e}\right); \tag{39}$$

here L is a constant and the number e may be replaced by any constant >1 at the price of a simple alteration of L.

For arbitrary real x let C_x be the part of the circumference of the circle with radius $|x|$ and center the origin, which contains no points of D; let

$$\Delta = \bigcup_{x \in \mathcal{C}E_h} (C_x \cup P),$$

P being the half-plane $x < -h$.

From the validity of (39) on $\mathcal{C}D$ it is immediately clear that on $\bar{\Delta}$ we have

$$|\Phi(z)| \leqq L_1 M\left(\frac{|r|}{e}\right), \qquad (L_1 \text{ constant}). \tag{40}$$

Taking $z = -ie^s$ we see that $F(s) = \Phi(-ie^s)$ is holomorphic in the domain

$$\Delta_s: \quad \sigma \geqq \sigma_0, \quad -G_2(\sigma) < t < G_1(\sigma), \qquad (\sigma_0 \text{ constant}),$$

where

$$G_1(\sigma) = \pi u(e^\sigma) - \arccos(he^{-\sigma}),$$

$$G_2(\sigma) = \pi u(-e^\sigma) - \arccos(he^{-\sigma}), \qquad (0 < \arccos he^{-\sigma} < \pi/2).$$

Furthermore, in Δ_s:

$$|F(s)| \leqq L_1 M(e^{\sigma-1}), \tag{41}$$

since Δ_s is contained in the map, under $z = -ie^s$ of Δ (repeated with the period $2\pi i$).

If, c being an appropriately chosen constant,

$$S(\sigma) = \pi \int_c^\sigma \frac{d\tau}{G(\tau)}, \qquad (G(\tau) = G_1(\tau) + G_2(\tau))$$

we have

$$S(\sigma) - \int_c^\sigma \frac{d\tau}{u(e^\tau) + u(-e^\tau)} = O(1), \qquad \text{as } \sigma \to \infty.$$

Hence (37) may also be written as

$$-\int^\infty \log M(e^{\sigma-1})e^{-S(\sigma)}d\sigma = \infty. \tag{42}$$

With (41) and (42) we can then apply Theorem 8a, which yields

$$F(s) \equiv 0 \quad \text{and hence} \quad \Phi(z) \equiv 0, \qquad \text{q.e.d.}$$

LEMMA M_4. *In Lemma* M_3 *the condition that* $u(x)$ *be of bounded variation may be replaced by the following hypothesis*:

$u(x)$ *is continuously differentiable,* $xu'(x)$ *is of bounded variation*[5] *and*

$$\int_{-\infty}^{\infty} |x|(u'(x))^2 dx < \infty.$$

Indeed, in this case $G_1(\sigma) = u(e^\sigma)$ and $G_2(\sigma) = u(-e^\sigma)$ have continuous derivatives which satisfy

$$\int^\infty |G_j'(\sigma)|^2 d\sigma < \infty \quad \text{and} \quad \int^\infty |dG_j(\sigma)| < \infty, \qquad (j = 1, 2).$$

We may then apply Theorem 8b in the same way as Theorem 8a was used to obtain the conclusion of Lemma M_3.

III. CLOSURE THEOREMS

§1. Simple Closure Theorems

Having disposed of the indispensable preliminaries we are now in a position to enter into the subject with which we are principally concerned.

In this paragraph we shall operate in the function space $L_p(-\infty, \infty)$, designated as L_p. Convergence in L_p and the norm in L_p were discussed in Chapter I. Most integrals which arise being taken over the entire real axis, there is no risk of confusion in writing $\int$ for $\int_{-\infty}^{+\infty}$.

The *translations* $f(x+\xi)$, ξ constant, of $f(x) \in L_p$, $(1 \leqq p < \infty)$ span a closed linear manifold $\tau(f)$. In accordance with §3, Chapter I, $\phi(x) \in \tau(f)$ means that $\phi(x)$ is the limit in L_p of linear combinations of translations of f; i.e., to every $\epsilon > 0$ corresponds a finite integer $n(\epsilon)$ such that

$$\left\| \phi(x) - \sum_{\nu=1}^{n} a_{\nu,n} f(x + \xi_{\nu,n}) \right\| < \epsilon,$$

where $\xi_{1,n}, \xi_{2,n}, \cdots, \xi_{n,n}$ and $a_{1,n}, a_{2,n}, \cdots, a_{n,n}$ are constants the ξ being real, and $\| \ \|$ is the norm in L_p.

The following theorem, now classical, is due to N. Wiener [21].

THEOREM W. *A necessary and sufficient condition that $\tau(f) \equiv L_1$, where $f \in L_1$, is that the set $\Omega(f)$ of the zeros of the Fourier transforms $S(f)$ of f, be void.*

The sufficiency of this condition follows from Theorem R and Carleman's lemma.

In the general case, when $\Omega(f)$ is not assumed empty, a necessary condition that $\psi \in \tau(f)$ is that $\Omega(\psi) \supset \Omega(f)$, where ψ and $f \in L_1$. S. Agmon and the author, in a joint paper, have given conditions which, with $\Omega(\psi) \supset \Omega(f)$, suffice to ensure that $\psi \in \tau(f)$ for ψ and f both in L_1, [14]. These conditions bear either on the intersection of the boundaries of the sets $\Omega(\psi)$ and $\Omega(f)$ (e.g. it is enumerable. cf. [14, p. 7], or, for instance, on the rate of growth of ψ, $(x\psi \in L_1)$.

Now, let f be an infinitely differentiable function on the entire real axis (for short: $f \in C^\infty$) and let $f^{(n)}(x) \in L_p$, $(n \geqq 0)$. Given a sequence $\{M_n\}$ of non-negative numbers, we say that $f \in L_p\{M_n\}$ if $\|f^{(n)}\| \leqq M_n$, $(n \geqq 0)$.

The sequence of functions $\{f^{(n)}(x)\}$ then generates a closed linear manifold $\delta(f)$ in L_p. Every function $\psi(x) \in \delta(f)$ is the limit of linear combinations of a finite number of $f^{(n)}(x)$; i.e., to every $\epsilon > 0$ corresponds a finite integer $n(\epsilon)$ such that, in L_p,

$$\left\| \psi(x) - \sum_{\nu=0}^{n} b_{\nu,n} f^{(\nu)}(x) \right\| < \epsilon,$$

the $b_{\nu,n}$ being constants.

The following lemma is immediate:

LEMMA 1. *If* $f^{(n)} \in L_p$, $(0 \leqq n \leqq m)$, *then* $f^{(n)} \in \tau(f)$, $(0 \leqq n \leqq m)$. *In particular when* $f^{(n)} \in L_p$, $(n \geqq 0)$, *then* $\delta(f) \subset \tau(f)$.

Following Theorems 5 and 6 of Chapter I, the linear functionals L_p are in one to one correspondence with the functions L_q, $(1/p+1/q=1$ when $1<p<\infty$, $q=\infty$ when $p=1)$. By Theorem R this lemma will be proven if we show that all functionals on L_p which are orthogonal to $\tau(f)$ are also orthogonal to $f^{(n)}$.

Assume then that $h(x) \in L_q$ satisfies

$$f * h = 0. \tag{1}$$

Since

$$\frac{d^n}{dx^n}(f * h) = \frac{d^n f}{dx^n} * h = 0,$$

we have the result.

The converse statement, however, is not necessarily true. Although $f \in C^\infty$ and $f^{(n)} \in L_p$ for all n, one cannot, without further hypotheses, be certain that for every real constant a, one has

$$f(x + a) \in \delta(f). \tag{2}$$

After establishing some results we shall be able to characterize certain classes $L_p\{M_n\}$ for which $\delta(f) \supset \tau(f)$.

LEMMA 2. *Let $f \in L_p$ and $f' \in L_p$; then, for every real constant a:*

$$f(x + a) - f(x) \in \tau(f').$$

The principle of the proof is the same as that of the previous lemma. Let $h \in L_q$, $(1/p + 1/q = 1)$ and $f' * h = 0$. Then

$$f * h = c,$$

where c is constant. Consequently, for every constant a we have

$$[f(x + a) - f(x)] * h(x) = 0. \qquad \text{q.e.d.}$$

The classes $L_p\{M_n\}$ for which $\delta(f) = \tau(f)$ are characterized by:

THEOREM I. *Given the positive sequence $\{M_n\}$ and setting*

$$C(\sigma) = \underset{n \geqq 0}{\text{l.u.b.}} (n\sigma - \log M_n),$$

a necessary and sufficient condition for (2) to be true, for every function $f \in L_p\{M_n\}$ and for every real constant a, is

$$\int^{\infty} C(\sigma) e^{-\sigma} d\sigma = \infty. \tag{3}$$

The necessity of this condition (3) is to be taken in the sense that any class $L_p\{M_n\}$ for which

$$\int^{\infty} C(\sigma) e^{-\sigma} d\sigma < \infty, \tag{4}$$

contains a function $f(x)$ such that for some real constant a, $f(x+a) \notin \delta(f)$. In view of Lemma 2 the sufficiency of condition (3) for the conclusion of this theorem derives from the following proposition which adds an interesting precision to our assertion.

THEOREM II. *If condition (3) is satisfied, we have $\tau(f') \subset \delta(f'')$.*

Again Theorem R and Theorems 5 and 6 of Chapter I come into play. Let $h(x) \in L_q$, $(1/p + 1/q = 1)$, be such that

$$\int f^{(n)}(y)h(y)dy = 0, \qquad (n \geqq 2). \tag{5}$$

It then follows that for every real a

$$\int f'(y+a)h(y)dy = 0. \tag{6}$$

Indeed, write

$$A(x) = \int f(x+y)h(y)dy,$$

so $A(x) \in C^\infty$ because $f \in C^\infty$ and, applying Hölder's inequality (where $(\int |h|^q)^{1/q}$ is replaced by ess. up. bd. $|h|$ when $q=\infty$, $p=1$) we obtain

$$|A^{(n)}(x)| \leqq \left(\int |f^{(n)}(x)|^p dx\right)^{1/p} \left(\int |h(x)|^q dx\right)^{1/q}$$
$$= kM_n, \qquad (n \geqq 0).$$

(5) is equivalent to

$$A^{(n)}(0) = 0, \qquad (n \geqq 2).$$

Condition (3) (which may be written $\int^\infty \log T(r) r^{-2} dr = \infty$, where $T(r) = \text{l.u.b.}\ (r^n/M_n)$) is a necessary and sufficient condition for the class $C\{M_n\}$ to be quasi-analytic. It follows that $A'(x) \equiv c$, c constant. Thus $A(x) \equiv cx + d$ while $|A(x)| \leqq kM_0$, so $c=0$, and that proves (6).

We have shown that $\tau(f') \subset \delta(f'')$.

Lemma 2 states that $f(x \mp a) - f(x) \in \tau(f')$, so that finally we are assured that condition (3) ensures that, for every real constant a, $f(x+a)$ is the limit in L_p of linear combinations of the form $f(x) + \sum_2^\infty b_k f^{(k)}(x)$.

Now, let (4) be satisfied. We shall construct $f \in L_p\{M_n\}$ for which $f(x+a) \notin \delta(f)$ for some a. Now (4) gives $\lim M_n^{1/n} = \infty$, (see [9]). Designate by $\{\log M_n^c\}$ the convex regularized sequence of $\{\log M_n\}$. By changing, if necessary, at most a finite number of M_n we may make M_n^c increasing. Since we also have $C(\sigma) = \text{l.u.b.}_{n \geqq 0}\ (n\sigma - \log M_n^c)$, we know by (4) that there

is a not identically vanishing function $g(x)$ such that $g(x)=0$ for $x\notin[0, 1]$ and $|g^{(n)}(x)|\leqq M_{n-2}^{c}$, $(n\geqq 2)$.

Let $f(x)=g(x)-g''(x)$, so $|f^{(n)}(x)|\leqq 2M_n^c\leqq 2M_n$, $f(x)\not\equiv 0$, while $f(x)\equiv 0$ for $x\notin[0, 1]$. Let $G=S(g)$ and $F=S(f)$; then $F(u)=(1+u^2)G(u)$. Since[6] $1/(1+u^2)=S(\sqrt{2\pi}e^{-|x|})$ and, since $G(u)\in L$, we may apply relation (8) of Chapter I to obtain

$$G(u)=\frac{F(u)}{1+u^2}=S(f*e^{-|x|}).$$

In other terms

$$g(x)=\int f(x+y)e^{-|y|}dy.$$

Taking any a for which $g(a)\neq 0$ we have simultaneously

$$g(a)=\int f(x+a)e^{-|x|}dx\neq 0$$

and

$$g^{(n)}(0)=\int f^{(n)}(x)e^{-|x|}dx=0.$$

It is plain, by Theorem R, that $f(x+a)\notin\delta(f)$. Since $f(x)\in L_p$, $(1\leqq p\leqq\infty)$, our result is proved.

Remarks. 1) All the results of this paragraph extend to functions $\in C\{M_n\}$ (i.e., $f\in C^{\infty}$, $|f^{(n)}|\leqq k^nM_n$, $(n\geqq 0)$). Here, however, no convenient norm is available, the convergence relations hold in the sense that there is uniform convergence on every bounded interval $\subset R$. The proofs of the Lemmas can be made more direct. The proofs of Theorems I and II remain the same in principle, for Theorem R is valid in locally convex topological vector spaces, $h(x)dx$ is replaced by $d\mu(z)$, μ being a measure identically zero outside a finite interval. For properties of convex topological vector spaces see [18].

2) The manifold $\delta(f)$ may be replaced by a manifold generated by a sequence $\{f^{(n_\nu)}(x)\}$. This is achieved with the use of the author's theorems on generalized quasi-analyticity (see [11]).

§2. General Closure Theorems

It has just been shown that in order to be able to assert that $f(x+a)\in\delta(f)$, for every $f\in L_1\{M_n\}$ and for every real constant a, it is necessary that (3) be satisfied. On the other hand, in order to have $f(x+a)\in\tau(g)$ where f and $g\in L_1$, it is necessary that $\Omega(f)\supset\Omega(g)$. We shall now give conditions, bearing on both $\{M_n\}$ and the Fourier transform $S(g)=G$ which ensure the possibility of approximating $f(x+a)\in L_1$ with linear combinations of derivatives of f and of translations of g. This will require neither (3) nor the inclusion $\Omega(f)\supset\Omega(g)$; the requisite conditions, couched in explicit form, consist in the divergence of an integral resembling (3), but in which $e^{-\sigma}$ is replaced by a function which vanishes more slowly at infinity—the fuller the set $\Omega(g)$, the more rapidly this function must approach zero. Thus, if this set is "sufficiently lean," the function $C(\sigma)$ is allowed to be "sufficiently small"; i.e. the M_n are permitted to be "large."

Before stating these conditions in the form of a divergent integral we shall give less explicit but definitely more general conditions from which they follow on application of the results of Chapter II.

Given a closed set $E\subset R$, the real axis in the complex $\zeta=\xi+i\eta$ plane, we shall write

$$\Delta_E=\begin{cases}\text{the half-plane } \eta>0 \text{ when } E=R,\\ \text{the complex } \zeta \text{ plane minus the set } E \text{ when } E\neq R.\end{cases}$$

We recall then that a positive function $M(r)$, defined for $r>0$, is said to be a *function associated with* E if every holomorphic uniform function, $\Phi(\zeta)$ in Δ_E, which satisfies the inequality

$$|\Phi(\zeta)|\leqq\frac{M(|\zeta|)}{|\eta|}$$

in Δ_E, is identically zero.[7]

If $M(r)$ is associated with E_1 it is associated with every set $E_2\subset E_1$.

A being a set on the real axis R, $\mathcal{C}A=R-A$, $\mathfrak{I}A$ represents the interior of A. If A is closed it can be decomposed, by Cantor's Theorem, into a denumerable set $\mathcal{R}A$ and a perfect or void set

$\mathcal{P}A$ so that $A = \mathcal{R}A \cup \mathcal{P}A$. $\mathcal{P}A$ is the intersection of the derived sets, of all orders (finite and transfinite), of A.

We now give the general theorems of closure involving hypotheses on the associated functions of sets defined by the functions f and g. Once and for all, both $f^{(n)}$ *and* g *are taken to belong to* L_1.

THEOREM IIIa. *Let* f *be* p *times differentiable* ($p \geqq 0$, *finite or infinite*) *and let* $\|f^{(n)}\| \leqq M_n < \infty$, $(0 \leqq n \leqq p)$, *and let* $g \in L_1$. Let

$$M(r) = \underset{0 \leqq n < p+1}{\text{g.l.b.}} \frac{M_n}{r^n}. \tag{7}$$

Then, if $M(r)$ *is associated with the set* $\Omega(g)$, $f(x+a)$ *is, for any real constant* a, *the limit, in* L_1, *of linear combinations of the form*

$$\sum_{n \leqq N < p} a_{n,N} f^{(n)}(x) + \sum_{n \leqq M} b_{n,M} g(x + \xi_{n,M}). \tag{8}$$

In (8) $n \geqq 0$; so if $p = 0$ the first sum is not present and only $\tau(g)$ is involved.

We write $0 \leqq n < p+1$ rather than $0 \leqq n \leqq p$, since we desire to include the case, *most important of all*, corresponding to $p = \infty$.

THEOREM IIIb. *In* IIIa *the set* $\Omega(g)$ *can be replaced by* E_1, *where*

$$E_1 = \Omega(g) \cap \overline{\mathcal{C}\Omega(f)}. \tag{9}$$

THEOREM IIIc. *If* $\mathcal{R}E_1 \subset \Omega(f)$, *the set* E_1 *in Theorem IIIb can be replaced by* E_2, *where*

$$E_2 = \mathcal{P}E_1 \tag{10}$$

If $\mathcal{R}E_1 \subset \Omega(f)$, Theorem IIIc is more general than Theorem IIIb. Theorem IIIb is more general than IIIa. This follows from the inclusions:

$$E_2 \subset E_1 \subset \Omega(g).$$

Let us also note that Theorems IIIb and IIIc, both with p *positive* and *finite*, may be considered as corollaries of Theorem IIIc with $p = 0$. Indeed, $p > 0$ and finite, $M(r)$ is of the form c/r^p, for r large. But such a function cannot be associated with

a set containing more than p points, since for any given set of $p+1$ real points there exists a meromorphic function $\Phi(\zeta)\not\equiv 0$, with simple poles at these points and such that $|\Phi(\zeta)| \leqq M(|\zeta|)/|\eta|$. Thus if IIIb is satisfied with $p>0$ *finite*, E_1 has at most p points, say $u_1, \cdots, u_p$, and, a being any real constant, it is possible to determine constants $a_0, \cdots, a_{p-1}$ such that, on setting

$$f_1(x) = f(x+a) - \sum_0^{p-1} a_n f^{(n)}(x),$$

$$S(f_1) = \left(e^{iau} - \sum_0^{p-1} a_n i^n u^n\right) S(f),$$

we have $u_k \in \Omega(f_1)$, $(k=1, \cdots, p)$, and the hypotheses of IIIc are satisfied with f replaced by f_1 and $p=0$; that is to say $f_1 \in \tau(g)$. If now the hypotheses of IIIc with $p>0$, finite, are satisfied, then E_2 has a finite number of points. But $\mathcal{P}E_1 = E_1 \cap \mathcal{P}E_1 \subset E_2$, and thus $\mathcal{P}E_1$ (being perfect) is empty, that is to say $E_1 = \mathcal{R}E_1$, and E_2 is empty. This shows that the hypotheses of IIIc with $p=0$ are satisfied and $f \in \tau(g)$.

In view of Theorems 5 and R of Chapter I, Theorems IIIb, IIIc follow from:

Lemma III. *Under the hypotheses of Theorems IIIb or IIIc every function $h \in L_\infty$ which satisfies the relations*

$$\int f^{(n)}(-y)h(y)dy = 0, \qquad (0 \leqq n < p)^8 \tag{11}$$

$$\int g(x-y)h(y)dy = g * h = 0, \tag{12}$$

also satisfies

$$\int f(x-y)h(y)dy = f * h = 0. \tag{13}$$

In other words: *under the hypotheses of Theorems IIIb or IIIc every continuous linear functional on L_1 which is orthogonal to $f^{(n)}(-x)$, $(0 \leqq n < p)$, and to $\tau(g)$ is also orthogonal to $\tau(f)$.*

Proof: a) We first prove the lemma under the hypotheses of Theorem IIIb. $M(r)$ being the function associated with the set $E_1 = \Omega(g) \cap \overline{\mathcal{C}\Omega(f)}$.

Consider the bounded, continuous, p times differentiable function

$$p(x) = f * h. \tag{14}$$

Then

$$p^{(n)}(x) = f^{(n)} * h, \qquad (0 \leqq n < p + 1), \tag{15}$$

so the relations(11) express the fact that

$$p^{(n)}(0) = 0, \qquad (0 \leqq n < p). \tag{16}$$

Also, by (15) and the hypotheses,

$$\left| p^{(n)}(x) \right| \leqq kM_n, \qquad (0 \leqq n < p + 1). \tag{17}$$

Combining(12) and (14) gives

$$g * p = 0. \tag{18}$$

The function

$$\Phi(\zeta) = \int_{-\infty}^{0} e^{-ix\zeta} p(x)dx \tag{19}$$

is holomorphic in $\eta > 0$. Carleman's lemma and the lemma of Agmon and the author given in Chapter I, assert that if $E_1 \neq R$ the function $\Phi(\zeta)$ can be continued analytically into the lower half-plane, $\eta < 0$, across the set $\mathcal{C}E_1$ of the real axis R because

$$\mathcal{C}E_1 = \mathfrak{I}\Omega(f) \cup \mathcal{C}\Omega(g).$$

Furthermore the continued function is holomorphic and single-valued in Δ_{E_1}, its value in the lower half-plane being

$$\Phi(\zeta) = -\int_{0}^{\infty} e^{-ix\zeta} p(x)dx \qquad (\eta < 0). \tag{20}$$

On integrating (19) by parts n times $(0 \leqq n < p + 1)$ we have

$$\Phi(\zeta) = \frac{(-i)^n}{\zeta^n} \int_{-\infty}^{0} e^{-ix\zeta} p^{(n)}(x)dx, \qquad (\eta > 0), \tag{21}$$

and it follows from (17) that

$$|\Phi(\zeta)| \leqq \frac{kM_n}{|\eta\zeta^n|}, \qquad (0 \leqq n < p+1). \tag{22}$$

Taking (20) into account as well, we see that, if $E_1 \neq R$, (22) is true throughout Δ_{E_1}. Thus we have, in Δ_{E_1}:

$$|\Phi(\zeta)| \leqq k\frac{M(|\zeta|)}{|\eta|}. \tag{23}$$

Since $M(r)$ is associated with E_1, it follows that we must have $\Phi(\zeta) \equiv 0$. Note that when $E_1 = R$ the functions defined in (19) and (20), which are no longer necessarily continuations of each other, are both identically zero as a result of our definition of the function $M(r)$ associated with R.

We have shown that, under the hypotheses stated, $\Phi(\zeta) \equiv 0$ and so, by (14), (13) must be true.

b) Next, we extend the proof to hold under the hypotheses that $\mathcal{R}E_1 \subset \Omega(f)$ and that $M(r)$ is associated with the set $E_2 = E_1 \cap \mathcal{C}(\mathcal{R}E_1)$.

In part a) of the proof we have seen that $\Phi(\zeta)$ is regular throughout the set $\mathcal{C}E_1$. Whatever singularities $\Phi(\zeta)$ may have must then belong to the set E_1; we shall show that they must belong to a perfect set, hence $\Phi(\zeta)$ will be analytic in $\mathcal{R}E_1$ and its singularities will be confined to the set $E_2 = E_1 \cap \mathcal{C}(\mathcal{R}E_1)$. Specifically, we prove that no point $\xi_1 \in \Omega(f)$ can be an isolated singularity of $\Phi(\zeta)$ then (since $\mathcal{R}E_1$ is assumed $\subset \Omega(f)$, it follows by transfinite induction that no point of $\mathcal{R}E_1$ is singular, since $\mathcal{R}E_1$ consists of the isolated points of the successive derived sets of E_1. The proof which follows was given in [14].

Let us assume, then, that there exists a point $\xi_1 \in \Omega(f)$ which is an isolated singularity of $\Phi(\zeta)$. At worst ξ_1 can only be a simple pole, for (22) gives

$$\Phi(\zeta) < \frac{kM_0}{|\eta|},$$

and from this it is a simple matter to show that in a sufficiently

small neighborhood of ξ_1, $\Phi(\zeta)$ is majorized by $A|\xi_1-\zeta|^{-1}$, A constant.

ξ_1 being isolated, there is an $l_0>0$ such that, in $|\xi_1-\zeta|\leqq l_0$, $\Phi(\zeta)$ has no other singularities than ξ_1; take $0<l<l_0$ and let a be the residue at ξ_1 so that

$$\Phi(\zeta) = \frac{a}{\zeta - \xi_1} + \Phi_1(\zeta), \tag{24}$$

where $\Phi_1(\zeta)$ is holomorphic in $|\zeta-\xi_1|\leqq l_0$.

Let

$$\Delta(x) = \begin{cases} 1-|x| & \text{for } |x|\leqq 1, \\ 0 & \text{for } |x|>1. \end{cases}$$

When $\eta>0$ we have

$$\begin{aligned} &\int_{\xi_1-l}^{\xi_1+l} \Delta\left(\frac{\xi-\xi_1}{l}\right)[\Phi(\xi+i\eta)-\Phi(\xi-i\eta)]d\xi \\ &\quad = \int_{\xi_1-l}^{\xi_1+l} \Delta\left(\frac{\xi-\xi_1}{l}\right)d\xi \int p(y)e^{-\eta|v|-i\xi v}dy \\ &\quad = \int p(y)e^{-\eta|v|}dy \int_{\xi_1-l}^{\xi_1+l} \Delta\left(\frac{\xi-\xi_1}{l}\right)e^{-i\xi v}d\xi \\ &\quad = l\sqrt{2\pi}\int e^{-\eta|v|}p(y)e^{-i\xi_1 v}\delta(ly)dy, \end{aligned} \tag{25}$$

while $S(\delta)=\Delta$ (cf. Note 1).

Now $p(t)\delta(lt)\in L_1$, and we may use the Theorem of Lebesgue to obtain

$$\begin{aligned} &\lim_{\eta\to 0}\int_{\xi_1-l}^{\xi_1+l} \Delta\left(\frac{\xi-\xi_1}{l}\right)[\Phi(\xi+i\eta)-\Phi(\xi-i\eta)]d\xi \\ &\qquad = l\sqrt{2\pi}\int p(y)e^{-i\xi_1 v}\delta(ly)dy. \end{aligned}$$

Substituting (24) in this yields

$$l\int p(y)e^{-i\xi_1 y}\delta(ly)dy$$

$$= -\lim_{\eta\to 0}\frac{1}{\sqrt{2\pi}}\int_{\xi_1-l}^{\xi_1+l}\Delta\left(\frac{\xi-\xi_1}{l}\right)\frac{2ai\eta}{(\xi-\xi_1)^2+\eta^2}d\xi \tag{26}$$

$$= -\frac{2ai}{\sqrt{2\pi}}\int\frac{d\xi}{1+\xi^2} = -ai\sqrt{2\pi}.$$

But $\xi_1\in\Omega(f)$, so, $(F=S(f))$,

$$\int\delta(lx)e^{-i\xi_1 y}f(y-x)dy = \delta(lx)e^{-i\xi_1 x}F(\xi_1) = 0. \tag{27}$$

Combining (14) and (27) yields

$$l\int p(y)e^{-i\xi_1 y}\delta(ly)dy$$

$$= l\int e^{-i\xi_1 y}\delta(ly)dy\int f(y-x)h(x)dx$$

$$= l\int h(x)dx\int e^{-i\xi_1 y}\delta(ly)f(y-x)dy \tag{28}$$

$$= l\int h(x)dx\int e^{-i\xi_1 y}[\delta(ly)-\delta(lx)]f(y-x)dy,$$

and it follows on comparing (26) and (28) that ($m=$ess. up. bd. $|h(x)|$)

$$\sqrt{2\pi}\,|a| \leqq ml\int dx\int|\delta(ly)-\delta(lx)|\cdot|f(y-x)|\,dy$$

$$= m\int|f(y)|\,dy\int|\delta(t+ly)-\delta(t)|\,dt,\ (0<l<l_0).$$

We need merely let $l\to 0$ to see that $a=0$ and hence that Φ is analytic at ξ_1, a result contradicting the original assumption concerning ξ_1. All the relations derived in part a of the proof consequently hold in ζ plane minus the set E_2. q.e.d.

Remark. The present proof is valid only in L_1.

Let us examine some particular cases of Theorem III. We use results concerning the function $M(r)$ associated with some special sets E which were described in Chapter II.

It was shown that $M(r)=M=$*constant* $(r>0)$ *is associated with the empty set.* Thus if, *in Theorem IIIa, we take* $p=0$ *and* $\Omega(g)=\Lambda$, *the empty set, we conclude that every function of* L_1 *belongs to* $\tau(g)$ *when* $\Omega(g)=0$. *This is Theorem W* which is seen to be a particular case of Theorem IIIa (and therefore of Theorem IIIb).

Next let E^* represent the boundary of the set E. Clearly, if

$$\Omega(g) \subset \Omega(f) \tag{30}$$

and if the set $I=\Omega^*(f)\cap\Omega^*(g)$ is enumerable, then $\mathcal{R}E_1\subset\Omega(f)$ and the set E_2 in the statement of IIIc is void. Indeed

$$E_1 = \overline{\mathcal{C}\Omega(f)} \cap \Omega(g),$$

and by (30),

$$E_1 = I \subset \Omega^*(f) \subset \Omega(f).$$

Since I is enumerable, $\mathcal{P}I$ is empty, and we may write $\mathcal{R}E_1 = \mathcal{R}I = I = E_1$, whence

$$E_2 = E_1 \cap \mathcal{C}\mathcal{R}E_1 = E_1 \cap \mathcal{C}E_1 = \Lambda.$$

We know that $M(r)=M=$constant is associated with the empty set and so conclude from IIIc that:

If $\Omega(g)\subset\Omega(f)$ *and if the set* $\Omega^*(f)\cap\Omega^*(g)$ *is enumerable, then* $f\in\tau(g)$.

This corollary to Theorem IIIc is a generalization of Theorem W since $\Omega(g)$ is no longer assumed empty. It was given by S. Agmon and the author [14].

Another interesting special case corresponds to the function $M(r)$ which is associated with the entire real axis R. We saw in Chapter II that a decreasing function $M(r)$ satisfying

$$-\int^{\infty} \frac{\log M(r)}{r^2}\, dr = \infty \tag{31}$$

is associated with the real axis. Such a decreasing function $M(r)$ is *a fortiori* associated with any set on the real axis. Setting

$C(\sigma) = -\log M(e^{\sigma})$ transforms (31) into (3). The function $M(r)$ defined in (7) is decreasing and, for $p = \infty$,

$$C(\sigma) = -\log M(e^{\sigma}) = \underset{n \geqq 0}{\text{l.u.b.}} (n\sigma - \log M_n)$$

so that Theorem IIIa, with $p = \infty$, $g \equiv 0$ yields the result that, under condition (3), $f(x+a) \in \delta(f)$. This is the sufficiency part of Theorem I in L_1.

At the beginning of this section it was announced that the Theorems III can be put in more explicit terms. Clearly any result actually exhibiting the function $M(r)$ associated with a set on the real axis can be injected into the statement of any one of Theorems IIIa, IIIb or IIIc to give an explicit condition connecting the norms M_n, of the derivatives of f, with the sets $\Omega(f)$ and $\Omega(g)$. In Theorem 2, Chapter II we exhibited a function associated with general sets on R. Combined with Theorem IIIb it gives the important result:

THEOREM IV. *Given* $f \in L_1\{M_n\}$ *and* $g \in L_1$, *let*

$$C(\sigma) = \underset{n \geqq 0}{\text{l.u.b.}} (n\sigma - \log M_n),$$

and

$$E_1 = \Omega(g) \cap \overline{\mathcal{C}\Omega(f)}. \tag{32}$$

Let $\phi_1(x)$ *be the characteristic function of the set* $\mathcal{C}(E_{1,h})$ *where* h *is a positive constant.* ($E_{1,h}$ *is the* h*-covering of* E_1.)

If there exists a continuous function $u(x)$, $(-\infty < x < +\infty)$ *with the following properties*:

1) *There exist two positive constants* A *and* B *such that*

$$A \leqq u(x) \leqq \tfrac{1}{2} + \phi_1(x) + B\phi_1(x)\phi_1(-x), \tag{33}$$

2) $u(x)$ *is of bounded variation,*

3) $$\int^{\infty} C(\sigma) \exp\left[-\int^{\sigma} \frac{d\tau}{u(e^{\tau}) + u(-e^{\tau})}\right] d\sigma = \infty,$$

then, for every real constant a, $f(x+a)$ *is the limit, in* L_1, *of linear combinations of the form*

$$\sum_{n=0}^{N} (a_{n,N} f^{(n)}(x) + b_{n,N} g(x + \xi_{n,N})). \tag{34}$$

Remarks. 1) If the modified form of Theorem 3, Chapter II, mentioned in the remark following that theorem, is used, condition 2 above is changed to: *$u(x)$ has a continuous derivative $u'(x)$ such that $xu'(x)$ is of bounded variation and such that*

$$\int |x|(u'(x))^2 dx < \infty.$$

2) *If $\mathcal{R}E_1 \subset \Omega(f)$, condition* (33) *can be replaced by the condition*

$$A \leqq u(x) \leqq \tfrac{1}{2} + \phi_2(x) + B\phi_2(x)\phi_2(-x), \tag{35}$$

where $\phi_2(x)$ is the characteristic function of $\mathcal{C}(E_{2,h})$ and $E_2 = \mathcal{P}E_1$.

3) *Condition* (33) *can naturally be replaced by the more restrictive condition*

$$A \leqq u(x) \leqq \tfrac{1}{2} + \phi_0(x) + B\phi_0(x)\phi_0(-x), \tag{36}$$

where $\phi_0(x)$ is the characteristic function of $\mathcal{C}(\Omega_h(g))$.

When one of the sets $\Omega(g)$ or E_j, $(j=1, 2)$ is taken to be symmetrical Theorem IV assumes a simpler form:

THEOREM IVs. *f, g, $\{M_n\}$ and $\mathcal{C}(\sigma)$ being defined as in Theorem IV, let E_1 be a symmetrical set ($-x \in E_1$ whenever $x \in E_1$) and let $\phi^*(x)$ be the characteristic function of $E_{1,h}$, where h is a positive constant.*

If there exists a bounded continuous function $a(x)$ such that:

1) $a(x) \geqq \phi^*(x)$, $a(x) \geqq b > 0$, b constant,

2) *either $a(x)$ is of bounded variation, or $a(x)$ has a continuous derivative such that $xa'(x)$ is of bounded variation and*

$$\int |x|(a'(x))^2 dx < \infty,$$

and

$$3) \qquad \int^{\infty} C(\sigma) \exp\left[-\int^{\sigma} a(e^{\tau})d\tau\right] d\sigma = \infty,$$

then the conclusion of Theorem IV holds.

The conclusion also remains valid if $\Omega(g)$ is assumed to be sym-

metrical, providing E_h is replaced by $\Omega_h(g)$ in the statement. Again, when $\mathcal{R}E_1 \subset \Omega(f)$, E_2 being assumed symmetrical and $E_{1,h}$ being replaced by $E_{2,h}$ in the statement, the conclusion is still correct.

This theorem is a particular case of Theorem IV. Indeed we need merely observe that one may set $a(x) = 1/(2u(x)) = 1/(u(x)+u(-x))$, $(x \geqq x_0)$, and that $\phi^*(x) = 1-\phi(x)$.

If $E_j(\sigma)$, $(j=0, 1, 2)$ designates the map of the set E_j, $(E_0 = \Omega(g))$ under the transformation $x = e^\sigma (\sigma \in E_j(\sigma)$ when $x > 0$ and $x \in E_j)$, one sees that, when E_j is symmetrical, $\int_c^\sigma a(e^\tau)d\tau$, ($c$ arbitrary constant, $\sigma > c$) represents what might be called the "corrected measure" of the portion of $E_j(\sigma)$ between c and σ. It is natural also to call $C(\sigma)$ the "differentiability" of f. Without going into excess detail, one can say that condition 3, of Theorem IVs, measures the "total mean differentiability of f on E_j." A similar remark applies to Theorem IV.

IV. APPLICATIONS OF THE GENERAL CLOSURE THEOREMS

§1. Polynomial Approximation

Theorems III and IV of the previous chapter lead naturally to the following proposition.

THEOREM 1. *Under the hypotheses of Theorems IIIa, IIIb, IIIc or IV, IVs, every function* $\in\tau(f)$ *is the limit, in* L_1, *of linear combinations of the form*

$$\sum_{n\leqq N_1<p} a_{n,N_1}f^{(n)}(x) + \sum_{n\leqq N_2} b_{n,N_2}g(x+\xi_n);$$

(with the hypotheses of Theorem IV, $p=\infty$*).*

Let us assume that a given function $\psi\in\tau(f)\subset L_1$, and write $\Psi=S(\psi)$, $F=S(f)$, $G=S(g)$, while supposing the hypotheses of Theorem IV to be satisfied. To every $\epsilon>0$ there then correspond: an integer $N>0$, the real or complex constants $a_{1,N}, \cdots, a_{N,N}$, $b_{0,N}, \cdots, b_{N,N}$, and the real quantities $\xi_{1,N}, \cdots, \xi_{N,N}$ such that

$$\frac{1}{\sqrt{2\pi}}\left|\int\left\{\psi(x)e^{-ixu} - \sum_0^N a_{n,N}f^{(n)}(x)e^{-ixu} - \sum_0^N b_{n,N}g(x+\xi_{n,N})e^{-ixu}\right\}dx\right|$$

$$\leqq \frac{1}{\sqrt{2\pi}}\int\left|\psi(x) - \sum_0^N\left\{a_{n,N}f^{(n)}(x) + b_{n,N}g(x+\xi_{n,N})\right\}\right|dx < \epsilon.$$

In other terms:

$$(1)\qquad \left|\Psi(u) - F(u)\sum_0^N a_{n,N}i^nu^n - G(u)\sum_0^N b_{n,N}e^{i\xi_{n,N}u}\right| < \epsilon.$$

By definition $F(u)=0$ when $u\in\Omega(f)$ and $G(u)=0$ when $u\in\Omega(g)$. Thus we have proved:

THEOREM 2. *The hypotheses of Theorem IV imply, for every* $\epsilon>0$, *the existence of an integer* N *and sets of quantities* $\{a_{n,N}\}$,

$\{b_{n,N}\}$, $\{\xi_{n,N}\}$ *such that on the sets* $\Omega(g)$ *and* $\Omega(f)$ *one has respectively*:

$$\left|\Psi(u) - F(u)\sum_{0}^{N} a_{n,N}u^n\right| < \epsilon$$

and

$$\left|\Psi(u) - G(u)\sum_{0}^{N} b_{n,N}e^{i\xi_{n,N}u}\right| < \epsilon.$$

For every closed set $E \subset R$ one can easily construct a function $g \in L_1$ with the property $\Omega(g) = E$ (use series with small enough coefficients of triangular functions in $R - E$; the inverse Fourier transform of this series belongs to L_1 and will do). Consequently we may state a particularized form of Theorem 2:

THEOREM 3. *Let* $F = S(f)$, *where* $f \in L_1\{M_n\}$ *and let* $\Psi = S(\psi)$, *where* $\psi \in \tau(f)$. *Let* E *be a closed set* $\subset R$ *and take* $h > 0$.

If there exists a function $u(x)$ *on* R, *bounded and continuous on* R *and bounded below by a positive constant, and having the following properties*:

1) $u(x) \leqq 1/2$ for $x \in E_h$, $u(x) \leqq 3/2$ for $x \notin E_h$ and $-x \in E_h$,

2) *either* $u(x)$ *is of bounded variation, or else* $u(x)$ *has a continuous derivative such that* $xu'(x)$ *is of bounded variation and*

$$\int |x|(u'(x))^2 dx < \infty,$$

and 3) *if* $C(\sigma)$ *is defined by*

$$C(\sigma) = \underset{n \geqq 0}{\text{l.u.b.}}\ (n\sigma - \log M_n), \tag{2}$$

then

$$\int^{\infty} C(\sigma) \exp\left[-\int^{\sigma} \frac{d\tau}{u(e^{\tau}) + u(-e^{\tau})}\right] d\sigma = \infty, \tag{3}$$

then, to every $\epsilon > 0$, *corresponds a polynomial* $P(u)$ *such that, on* E:

$$|\Psi(u) - F(u)P(u)| < \epsilon. \tag{4}$$

If we make full use of Theorem 2 of this chapter (or of Theorem IV) we can find more general statements. This is achieved by using the sets E_1 and E_2 as defined in these theorems, in place of $\Omega(g) = E$.

In particular it follows from Theorem W that, when $\Omega(f) = \Lambda$ (the empty set, so $F(u) \neq 0$), then, all other conditions of Theorem 3 concerning F and E being satisfied, every function $\Psi \in A$ allows an approximation of the type (4). Now it is plain that any function continuous on R, for which $\lim_{|u|\to\infty} H(u) = 0$, is the uniform limit of a sequence of functions of class A (it suffices to truncate $H(u)$). Consequently we have:

THEOREM 4. *Let $F = S(f)$, where $f \in L_1\{M_n\}$, with $F(u) \neq 0$, and $C(\sigma)$ defined as in (2). Suppose there exists a function $u(x)$ satisfying the conditions imposed on $u(x)$ in Theorem 3, and let $H(u)$ be a function continuous on R and such that $\lim_{|u|\to\infty} H(u) = 0$.*

Then, to every $\epsilon > 0$ corresponds a polynomial $P(u)$ such that, on E,

$$| H(u) - F(u)P(u) | < \epsilon.$$

Let us now prove

LEMMA 1. *Let $F(u)$ be a continuous positive even function on R, such that $-\log F(u)$ is a convex function of $\log u$, $(u > 0)$, and such that for all n: $\lim_{|u|\to\infty} u^n F(u) = 0$ and also such that, for sufficiently large u: $u > u_0$, at the points where $F'(u)$ exists it satisfies*

$$| F'(u) | \leqq \alpha u^\beta, \qquad \alpha, \beta \text{ constants.} \tag{5}$$

Under these conditions $F(u)$ is the Fourier transform of a function f belonging to a class $L_1\{M_n\}$, the M_n being such that, when

$$C(\sigma) = \text{l.u.b.}_{n \geqq 1} (n\sigma - \log M_n),$$

there exist two constants, γ_1 and γ_2, for which, when σ is sufficiently large,

$$-\log F(e^\sigma) \leqq 2C(\sigma + \gamma_1) + \gamma_2\sigma. \tag{6}$$

Proof. The function

$$f(x) = \frac{1}{\sqrt{2\pi}} \int F(u) e^{iux} du \tag{7}$$

exists since $F \in L_1$, f is infinitely differentiable, and

$$f^{(n)}(x) = \frac{i^n}{\sqrt{2\pi}} \int F(u) u^n e^{iux} du. \tag{8}$$

This can be integrated by parts to give:

$$\begin{aligned} f^{(n)}(x) &= -\frac{i^{n-1}}{x\sqrt{2\pi}} \int (F(u)u^n)' e^{iux} du \\ &= -\frac{i^{n-1}}{x\sqrt{2\pi}} \int (F'(u)u^n + nu^{n-1}F(u)) e^{iux} du. \end{aligned} \tag{9}$$

Since $F'(u) < 0$ for $u > u_1$ it follows from (5) that $(F'(u))^2 \leqq -\alpha u^\beta F'(u)$ when $u > u_2 = \max(u_0, u_1)$. (9) gives

$$\begin{aligned} &\int |xf^{(n)}(x)|^2 dx \\ &\quad = \int |F'(u)u^n + nu^{n-1}F(u)|^2 du \\ &\quad \leqq -2\alpha \int_{u_2}^{\infty} F'(u) u^{2n+\beta} du + n^2 \int (F(u))^2 u^{2n-2} du + A_0^n, \end{aligned}$$

and from this, on integrating by parts, we derive

$$\begin{aligned} &\int |xf^{(n)}(x)|^2 dx \\ &\quad \leqq 2\alpha(2n+\beta) \int_{u_2}^{\infty} F(u) u^{2n+\beta-1} du + n^2 \int (F(u))^2 u^{2n-2} du + A_1^n \\ &\quad \leqq A_1^n + A_2^n \int_1^{\infty} F(u) u^{2(n+k)} du, \end{aligned}$$

where A_1, A_2 and k are positive constants.

Let us write

$$m_n^2 = \max_{u \geqq 1} F(u)u^{2n}.$$

Applying Schwarz's inequality we get:

$$\int |f^{(n)}(x)|\,dx$$

$$\leqq 2\left(\int_0^1 |f^{(n)}(x)|^2 dx\right)^{1/2} + 2\left(\int_1^\infty \frac{dx}{x^2}\int_1^\infty x^2 |f^{(n)}(x)|^2 dx\right)^{1/2}$$

$$\leq 2\left(\int_0^\infty |F(u)|^2 u^{2n} du\right)^{1/2} + 2\left(A_1^n + A_2^n \int_1^\infty F(u)u^{2(n+k)} du\right)^{1/2}$$

$$\leqq 2(A_3^n + m_{n+1}^2)^{1/2} + 2(A_4^n + A_2^n m_{n+k+1}^2)^{1/2}.$$

The m_n are increasing so this gives

$$\|f^{(n)}\| = \int |f^{(n)}(x)|\,dx \leqq B^n m_{n+k+1} = M_n,$$

where B is a constant. Thus $f \in L_1\{M_n\}$.

Set

$$A(t) = \text{l.u.b.}_{\sigma>0}\,(\sigma t + \tfrac{1}{2}\log F(e^\sigma)).$$

Clearly $A(n) = \log m_n$ and $-\lim \log F(e^\sigma)\sigma^{-1} = \infty$. Following a result given in [10] we may assert that $\lim_{t\to\infty} A(t)t^{-1} = \infty$ and that, for sufficiently large σ,

$$-\tfrac{1}{2}\log F(e^\sigma) = \text{l.u.b.}_{\sigma>0}\,(\sigma t - A(t)) = \sigma t(\sigma) - A(t(\sigma)).$$

$A(t)$ is increasing, so if, for sufficiently large σ, $n+k+1$ is the greatest integer less than $t(\sigma)$:

$$\begin{aligned} -\tfrac{1}{2}\log F(e^\sigma) &\leqq \sigma(n+k+2) - \log m_{n+k+1} \\ &= n\sigma - \log m_{n+k+1} - n\log B + \sigma(k+2) + n\log B \\ &= n(\sigma + \log B) - \log M_n + \sigma(k+2) \\ &\leqq C(\sigma + \log B) + \sigma(k+2), \end{aligned}$$

q.e.d.

Lemma 2. *Let $F(u)$ be a positive continuous even function, where $-\log F(u)$ is a convex function of $\log u$, $(u>0)$ such that*

$$(10) \quad -\int^{\infty} \log F(e^{\sigma}) \exp\left[-\int^{\sigma} \frac{d\tau}{u(e^{\tau}) + u(-e^{\tau})}\right] d\sigma = \infty,$$

$u(x)$ being a function as described in conditions 1 *and* 2 *of Theorem* 3. *Then a continuous positive even function $F_1(u)$, can be constructed so that $F = O(F_1)$ and $F_1(u)$ has the property*:

$F_1 = S(f_1)$ with $f_1 \in L_1\{M_n\}$, the M_n being such that the function

$$C(\sigma) = \text{l.u.b.}_{n \geqq 0} (n\sigma - \log M_n)$$

satisfies the relation

$$(11) \quad \int^{\infty} C(\sigma) \exp\left[-\int^{\sigma} \frac{d\tau}{u_1(e^{\tau}) + u_1(-e^{\tau})}\right] d\sigma = \infty,$$

$u_1(x)$ being a function as described in conditions 1 *and* 2 *of Theorem* 3.

Proof: Observe that, $-\log F(u)$ being a convex function of $\log u$, all the derivatives appearing in the following exposition are meaningful, in the sense that derivatives exist almost everywhere and $-uF'(u)/F(u)$ is increasing as one moves from left to right.

Define the function $F_1(u)$ by the equality

$$(12) \quad (-\log F_1(u))' = \min\left(-\frac{F'(u)}{F(u)}, u\right), \qquad (u > 0),$$

from which follows

$$(13) \quad \frac{d(-\log F_1(u))}{d(\log u)} = u(-\log F_1(u))' = \min\left(-u\frac{F'(u)}{F(u)}, u^2\right),$$

where the right-hand side is increasing.[9] Thus $-\log F_1(u)$ is a convex function of $\log u$. Furthermore an integration in (12) shows that $F = O(F_1)$. Also from (12) we obtain, for all sufficiently large u,

$$(14) \quad 0 \leqq -F_1'(u) \leqq uF_1(u).$$

Thus either one of the two following alternatives may occur:

α) $-\dfrac{F_1'(u)}{F_1(u)} < u$ for all sufficiently large u,

$\beta)$ $-\dfrac{F_1'(u_i)}{F_1(u_i)} = u_i$ for an indefinitely increasing sequence of u_i.

Let us first assume that $\beta)$ arises. We may assume $u_{i+1} - u_i \geqq 1$, (all i), and $-F_1'(u) > 0$ for $u > u_1$. Under these circumstances we may write:

$$
\begin{aligned}
-\int_{u_1}^{\infty} u \frac{F_1'(u)}{F_1(u)} \frac{du}{u^2} &\geqq -\sum_1^{\infty} \int_{u_i}^{u_i+1} u \frac{F_1'(u)}{F_1(u)} \frac{du}{u^2} \\
&\geqq \sum_1^{\infty} \frac{u_i^2}{(u_i+1)^2} = \infty,
\end{aligned} \tag{15}
$$

which, changing variables, shows that

$$
-\int^{\infty} (\log F_1(e^{\sigma}))' e^{-\sigma} d\sigma = \infty. \tag{16}
$$

On the other hand, integrating by parts yields

$$
\begin{aligned}
-\int_{\alpha}^{N} \log F_1(e^{\sigma}) e^{-\sigma} d\sigma &= \log F_1(e^N) e^{-N} - \log F_1(e^{\alpha}) e^{-\alpha} \\
&\quad - \int_{\alpha}^{N} (\log F_1(e^{\sigma}))' e^{-\sigma} d\sigma.
\end{aligned} \tag{17}
$$

If the left-hand side in (17) converged as $N \to \infty$, there would exist a sequence $N_i \uparrow \infty$ for which $\log F_1(e^{N_i}) e^{-N_i} \to 0$, therefore, from

$$
\begin{aligned}
-\int_{\alpha}^{N_i} \log F_1(e^{\sigma}) e^{-\sigma} d\sigma &= \log F_1(e^{N_i}) e^{-N_i} - \log F_1(e^{\alpha}) e^{-\alpha} \\
&\quad - \int_{\alpha}^{N_i} (\log F_1(e^{\sigma}))' e^{-\sigma} d\sigma,
\end{aligned} \tag{18}
$$

it would follow that the integral in (16) would also converge, which contradicts relation (16). Consequently

$$
-\int^{\infty} \log F_1(e^{\sigma}) e^{-\sigma} d\sigma = \infty. \tag{19}
$$

Next, in the case of alternative α), it follows from the defining relation (12) that $-F_1'(u)/F_1(u) = -F'(u)/F(u)$ for all sufficiently large u and therefore, by (10),

$$(20) \qquad -\int^{\infty} \log F_1(e^{\sigma}) \exp\left[-\int^{\sigma} \frac{d\tau}{u(e^{\tau}) + u(-e^{\tau})}\right] d\sigma = \infty.$$

Thus, in any case, the function F_1 satisfies the relation

$$(21) \qquad -\int^{\infty} \log F_1(e^{\sigma}) \exp\left[-\int^{\sigma} \frac{d\tau}{u_1(e^{\tau}) + u_1(-e^{\tau})}\right] d\sigma = \infty,$$

in which $u_1(x) = u(x)$ in case α), and $u_1(x) \equiv 1/2$ in the case of alternative β).

From this, and the fact that $-\log F_1(u)$ is convex in $\log u$, it follows that $-\log F_1(e^{\sigma})/\sigma \to +\infty$ as $\sigma \to +\infty$; indeed, otherwise we should have $-\log F_1(e^{\sigma_i})/\sigma_i < K$ for $\sigma_i \uparrow \infty$, and the convexity property then makes

$$-\log F_1(e^{\sigma})/\sigma < K$$

for all σ large so that (21) would converge. Thus we see that for every constant K, no matter how large, we have

$$(22) \qquad u^K F_1(u) \to 0,$$

so that by (14), for u sufficiently large,

$$(23) \qquad |F_1'(u)| < \alpha u.$$

$F_1(u)$ has been shown to have all the properties of the function $F(u)$ described in Lemma 1 which is then applied to it to yield the conclusions of the present proposition (with $u_1(x) \equiv 1/2$ in the case of alternative β). q.e.d.

Lemmas 1 and 2 enable us to pass from Theorem 4 to:

THEOREM 5. *Let $F(u)$ be a positive, even, continuous function with $-\log F(u)$ convex in $\log u$, $(u > 0)$. Let $E \subset R$ be a closed set and let $u(x)$ be a continuous bounded function on R, satisfying conditions 1 and 2 of Theorem 3, bounded positively below, and such that*

$$(10) \qquad -\int^{\infty} \log F(e^{\sigma}) \exp\left[-\int^{\sigma} \frac{d\tau}{u(e^{\tau}) + u(-e^{\tau})}\right] d\sigma = \infty.$$

Let $H(u)$ be any continuous function on R, such that $\lim_{|u|\to\infty} H(u) = 0$.

Then, to every $\epsilon > 0$ corresponds a polynomial $P(u)$ such that, on E:

$$(24) \qquad |H(u) - F(u)P(u)| < \epsilon.$$

Proof: Let us construct $F_1(u)$ as described in Lemma 2. $F_1(u)$ satisfies all the requirements of Theorem 4 (with $u_1(x) \equiv u(x)$ in alternative α and $u_1(x) \equiv 1/2$ in case β) and so we may apply Theorem 4. Recalling that $F(u) = O(F_1(u))$, we shall now show that the result automatically remains true with $F(u)$ in (24).

For u sufficiently large, $|u| > u_c$, $|H(u)| < \epsilon$. Take

$$H_1(u) = \begin{cases} H(u) & \text{for } |u| \leqq u_c \\ \text{linear} & \text{for } u_c \leqq |u| \leqq u_c + 1 \\ 0 & \text{for } |u| \geqq u_c + 1, \end{cases}$$

and

$$H_2(u) = H(u) - H_1(u),$$

so that $H_2(u) \equiv 0$ in $|u| \leqq u_c$ and $|H_2(u)| < 2\epsilon$. Thus $H(u)$ is the sum of two continuous functions. $H_1(u)$ being null outside the finite interval $[-u_c, u_c]$, a polynomial $P(u)$ can be found for which

$$\left| H_1(u) \frac{F_1(u)}{F(u)} - P(u)F_1(u) \right| < \epsilon,$$

so

$$|H_1(u) - P(u)F(u)| < \frac{F(u)}{F_1(u)} \epsilon < k\epsilon, \qquad k \text{ constant.}$$

Therefore

$$|H(u) - P(u)F(u)| \leqq (k+2)\epsilon. \qquad \text{q.e.d.}$$

The above theorem contains the following particular case:

THEOREM 5′. *Let* $\Phi(u)$ *be a positive, even, continuous function with* $\log \Phi(u)$ *a convex function of* $\log u$, $(u>0)$.

If

$$\int^{\infty} \log \Phi(e^{\sigma})e^{-\sigma}d\sigma = \infty$$

(*or, equivalently,*

$$(\alpha) \qquad \int^{\infty} \frac{\log \Phi(u)}{u^2}\,du = \infty\Big),$$

then to every function $H(u)$, *continuous on* R *with* $\lim_{|u|\to\infty} H(u) = 0$ *corresponds a polynomial* $P(u)$ *such that, on* R,

$$\left| H(u) - P(u)/\Phi(u) \right| < \epsilon.$$

This result, with the restriction $\Phi'(u) = O(\Phi^2(u))$ also follows from a very general theorem of the author's [13]. Horvath [8], while using an inequality due to the author, was able to eliminate this restriction and stated the theorem in the form just given. Ahiezer and Babenko [2] derived a result which is apparently more general: the convexity of $\log \Phi(u)$ is not required; they merely suppose $\Phi(u) \geqq 1$ and replace condition (α) by the hypothesis that there exists a sequence of polynomials $P_{n_k}(u)$ with $\Phi(u) \geqq P_{n_k}(u)$ and for which

$$\int \frac{\log P_{n_k}(u)}{1+u^2}\,du = \infty,$$

($\Phi(u)$ is not assumed to be even).

Theorem 5′ contains the classical theorem of S. Bernstein in which it is supposed that

$$\Phi(u) = \prod_1^{\infty}\left(1+\frac{u^2}{\beta_n}\right), \qquad \sum \beta_n^{-2} < \infty,$$

$$\int^{\infty} \frac{\log \Phi(u)}{u^2}\,du = \infty.$$

That Theorem 5′ follows from Theorem 5 is obvious; in the case when $E=R$ it suffices to take $u(x) \equiv 1/2$.

§2. The Generalized Moment Problem

Let us recall some definitions (see [19]). A set function $V(e)$ is called a distribution function if it is non-negative, defined and finite on all Borel sets e on R, and if it is completely additive; i.e. if for every sequence $\{e_\nu\}$ of disjoint sets

$$V\left(\bigcup_{\nu\geqq 1} e_\nu\right) = \sum_1^\infty V(e_\nu).$$

The *spectrum*, $\Sigma(V)$, of the distribution function V, is the set of all points $x\in R$ such that $V(n_x)>0$ for every neighborhood n_x of x.

Given a sequence $\{m_\nu\}$ of real numbers and a closed set $E\subset R$, the "E-moment problem" consists in the determination of a distribution function $V(e)$, whose spectrum $\Sigma(V)\subset E$, and for which:

$$\int u^n dV = m_n, \qquad (n \geqq 0). \tag{25}$$

Two distribution functions V_1 and V_2 are said to be substantially equal if for every continuous function $f(u)$ on R, which is $=0$ for sufficiently large $|u|$, one has

$$\int f(u)dV_1 = \int f(u)dV_2. \tag{26}$$

The uniqueness problem in the theory of E-moments consists in determining conditions on $\{m_\nu\}$ which ensure that two distribution functions, V_1 and V_2, whose spectra are contained in E and for which

$$\int u^n dV_1 = \int u^n dV_2 = m_n, \qquad (n \geqq 0), \tag{27}$$

are substantially equal. This problem has been solved for $E=R$ (Hamburger moment problem) and for $E=\{x: x\geqq 0\}$ (Stieltjes moment problem). We shall give a solution for a general $E\subset R$.

THEOREM 6. *Given the sequence of real numbers* $\{m_n\}$ *and the*

closed set $E \subset R$, suppose there exists a distribution function $V(e)$, whose spectrum is in E, $\Sigma(V) \subset E$, and which satisfies

$$\int u^n dV = m_n, \qquad (n \geqq 0). \tag{28}$$

Set: 1) $\Phi(u) = \sum \dfrac{u^{2n}}{m_{2n}}$, *if E contains points with negative abscissae.*

2) $\Phi(u) = \sum \dfrac{u^n}{m_n}$, *if all points of E have positive abscissae.*

Suppose there exists a function $u(x)$, bounded and continuous on R and bounded positively below, which satisfies conditions 1 *and* 2 *of Theorem* 3, *and for which*

$$\int^{\infty} \log \Phi(e^{\sigma}) \exp\left[-\int^{\sigma} \frac{d\tau}{u(e^{\tau}) + u(-e^{\tau})}\right] d\sigma = \infty. \tag{29}$$

Under these conditions any two distribution functions, whose spectra are in E and which satisfy (28), *are substantially equal.*

Proof: Suppose $\Sigma(V_1)$ and $\Sigma(V_2) \subset E$, and

$$\int u^n dV_1 = \int u^n dV_2 = m_n, \qquad (n \geqq 0). \tag{30}$$

This means that for every polynomial $P(u)$ we have, writing $W = V_1 - V_2$,

$$\int P(u) dW = 0. \tag{31}$$

From (30) we get

$$\int \Phi\left(\frac{t}{2}\right) dV_1 = \int \Phi\left(\frac{t}{2}\right) dV_2 \leqq \sum_0^{\infty} 2^{-n} = 2.$$

Let $F(u) = 1/\Phi(u/2)$ if E contains points with negative abscissae, $F(u) = 1/\Phi(|u|/2)$ otherwise. F satisfies the hypotheses of Theorem 5. If then $f(u)$ is a bounded continuous

function on R we find for every $\epsilon>0$ a polynomial $P(u)$ such that on E:

$$|f(u)F(u) - P(u)F(u)| < \epsilon. \tag{32}$$

Applying (31) to this polynomial gives

$$\int f(u)dW = \int (f(u) - P(u))dW, \tag{33}$$

and hence

$$\begin{aligned} \left|\int f(u)dW\right| &\leqq \int |f(u) - P(u)|\,|dW| \\ &\leqq \int |f(u) - P(u)|\,(dV_1 + dV_2) \\ &\leqq \epsilon \int \Phi\left(\frac{u}{2}\right)(dV_1 + dV_2) \leqq 4\epsilon. \end{aligned}$$

Therefore

$$\int f(u)dV_1 = \int f(u)dV_2, \qquad \text{q.e.d.}$$

Note that the transition from Theorem 5 to Theorem 6 is analogous to that used by S. Bernstein in the application of his theorem on approximation on R to the problem of Hamburger.

Theorem 6 includes as particular cases the theorems of Carleman [19] giving uniqueness conditions for the Stieltjes problem (spectrum on half-line) and the Hamburger problem (spectrum on R, or, rather, no conditions concerning the spectrum being given, $E=R$). Before quoting Carleman's theorems in their initial form we shall state them as follows:

THEOREM 7. (*Stieltjes problem*). *Let* $\{m_n\}$ *be a real sequence, and let there exist a distribution function* V, *whose spectrum is on the half-line* $x \geqq 0$, *and which satisfies*

$$\int u^n dV = m_n, \qquad (n \geqq 0). \tag{34}$$

If

$$\int^{\infty} \log \Phi(e^{\sigma}) e^{-\sigma/2} d\sigma = \infty, \tag{35}$$

where

$$\Phi(u) = \sum_{0}^{\infty} \frac{u^n}{m_n}, \tag{36}$$

then any two distribution functions, whose spectra are on $x \geqq 0$ *and which satisfy* (34), *are substantially equal.*

Indeed, let

$$u(x) = \begin{cases} \frac{1}{2} & \text{for } x \geqq 0 \\ \text{linear} & \text{for } x \in [-1, 0] \\ \frac{3}{2} & \text{for } x \leqq -1 \end{cases}$$

in Theorem 6.

THEOREM 8. (*Problem of Hamburger*). *Let* $\{m_n\}$ *be a real sequence and assume the existence of a distribution* V *for which*

$$\int u^n dV = m_n, \qquad (n \geqq 0). \tag{37}$$

If

$$\int^{\infty} \log \Phi(e^{\sigma}) e^{-\sigma} d\sigma = \infty, \tag{38}$$

where

$$\Phi(u) = \sum_{0}^{\infty} \frac{u^{2n}}{m_{2n}},$$

then any two distribution functions satisfying (37) *are substantially equal.*

This follows from Theorem 6 in which $u(x) \equiv 1/2$.

We now come to a lemma, of interest in itself, which will

lead from the last two theorems to the above-mentioned theorems of Carleman.

LEMMA. *Let* $V \not\equiv 0$ *be a distribution function with*

$$\int u^n dV = m_n. \tag{39}$$

Then $\log m_{2n}$ *is a convex function of* n; *if the spectrum of* V *is on the half line* $x \geqq 0$, $\log m_n$ *is convex in* n.

Take $c > 1/2$ and consider the polynomial

$$P_n(c, u) = c\left(\frac{m_{2(n+2)}}{m_{2n}}\right)^{1/2} u^{2n} - u^{2(n+1)} + c\left(\frac{m_{2n}}{m_{2(n+2)}}\right)^{1/2} u^{2(n+2)},$$

which is >0 for all values of u. From (39) we get

$$\begin{aligned} 0 &< \int P_n(c, u)dV \\ &= c\left(\frac{m_{2(n+2)}}{m_{2n}}\right)^{1/2} m_{2n} - m_{2(n+1)} + c\left(\frac{m_{2n}}{m_{2(n+2)}}\right)^{1/2} m_{2(n+2)} \\ &= 2c(m_{2n}m_{2(n+2)})^{1/2} - m_{2(n+1)}. \end{aligned}$$

So, letting $c \to 1/2$, this gives

$$m_{2(n+1)} \leqq (m_{2n}m_{2(n+2)})^{1/2},$$

which proves the first part of the lemma. To prove the second part one need only take

$$P_n(c, u) = c\left(\frac{m_{n+1}}{m_n}\right)^{1/2} u^n - u^{n+1} + c\left(\frac{m_n}{m_{n+1}}\right)^{1/2} u^{n+2}, \qquad (c > \tfrac{1}{2}),$$

and reason in the same way as before.

Observe that (35), with Φ defined in (36), is equivalent to

$$\int^{\infty} C(\sigma)e^{-\sigma/2}d\sigma = \infty, \tag{40}$$

where

$$C(\sigma) = \underset{n \geqq 0}{\text{l.u.b.}} (n\sigma - \log m_n), \tag{41}$$

($m_n \geqq 0$ for all n because the spectrum of V is on $x \geqq 0$).

This is seen as follows:

$$e^{C(\sigma)} = \text{l.u.b.}_{n \geqq 0} \frac{e^{n\sigma}}{m_n} \leqq \sum \frac{e^{n\sigma}}{m_n} = \Phi(e^{\sigma})$$

$$\leqq \text{l.u.b.}_{n \geqq 0} \frac{e^{n(\sigma+1)}}{m_n} \sum e^{-n} = a e^{C(\sigma+1)},$$

a being constant; thus (35) and (40) hold simultaneously. In the same way (38) with $\Phi(u) = \sum u^{2n}/m_{2n}$ is seen to be equivalent to (40) where

$$C(\sigma) = \text{l.u.b.}_{n \geqq 0} (n\sigma - \log m_{2n}). \tag{42}$$

But, $\log m_n$ being convex in n, (40) with $C(\sigma)$ defined by (41) is equivalent to (see [9])

$$\sum \left(\frac{m_n}{m_{n+1}}\right)^{1/2} = \infty. \tag{43}$$

Also we know that (43) is equivalent to the relation

$$\sum m_n^{-1/(2n)} = \infty.$$

Analogously (40), with $C(\sigma)$ defined in (42), holds simultaneously with the equivalent relations

$$\sum \left(\frac{m_{2n}}{m_{2(n+1)}}\right)^{1/2} = \infty \quad \text{and} \quad \sum \frac{1}{(m_{2n})^{1/2n}} = \infty.$$

These considerations prove the equivalence of Theorems 7 and 8 with the following statements:

THEOREM 7a. *If a Stieltjes problem has a solution this solution is unique when the following equivalent conditions are satisfied*:

$$\sum \left(\frac{m_n}{m_{n+1}}\right)^{1/2} = \infty, \qquad \sum (m_n)^{-1/2n} = \infty.$$

THEOREM 8a. *If a problem of Hamburger has a solution it is unique when the following equivalent conditions are satisfied*:

$$\sum\left(\frac{m_{2n}}{m_{2(n+1)}}\right)^{1/2} = \infty, \qquad \sum (m_{2n})^{-1/2n} = \infty.$$

Finally it is worth noting that, in their book [19], Shohat and Tamarkin thought it necessary to replace the conditions $\sum(m_n)^{-1/2n} = \infty$ and $\sum(m_{2n})^{-1/2n} = \infty$ respectively by other rather complicated conditions which they believed to be more general, viz. $\sum(\beta_n)^{-1/2} = \infty$, $\sum(\gamma_n)^{-1/2} = \infty$, where

$$\beta_n = \min_{h \geqq 0} (m_{n+h})^{1/(n+h)}, \qquad \gamma_n = \min_{h \geqq 0} (m_{2(n+h)})^{1/(n+h)}.$$

But the convexity ensures that for sufficiently large n: $(m_n)^{1/n} = \beta_n$ and $(m_{2n})^{1/n} = \gamma_n$ in the Stieltjes and Hamburger cases respectively; thus their result is not actually a generalization.

NOTES

1. Indeed $\Delta(x) \in L_1$ so we can apply the Fourier formula (5) to it and get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} \Delta(x) dx = \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x) \cos xt dx = \frac{2}{\sqrt{2\pi}} \frac{1 - \cos t}{t^2} = \delta(t).$$

   This is in L_1 so inversion (6) applies and must, of necessity, give back $\Delta(x)$.

2. Any zeros on the circumference will cause no trouble for we can find $r<1$ such that $\psi(re^{i\theta})$ has no zeros, and prove the theorem on this circumference, then letting $r \to 1$ afterwards does not destroy the inequalities.

3. The omission of a limit of integration in $\int^{\infty} C(x)dx$ simply means that $\int_a^{\infty} C(x)dx$ is meaningful and satisfies the indicated relation for all a sufficiently large. A similar remark applies to $\int_{-\infty}$.

4. In a paper to appear soon Mme. Lelong-Ferrand proves that in Theorem 9 the condition (a) $\int^{\infty} |dG_j'(\sigma)| < \infty$ can be replaced by the simpler condition (b) $\lim_{\sigma \to \infty} G_j'(\sigma) = 0$. Then of course in Theorem 8b, as well as in all the theorems where the condition (a) appears, (a) can be replaced by (b).

5. It follows from Note 4 that this condition, here and in the other theorems, can be replaced by $xu'(x) \to 0$, as $x \to \pm \infty$.

6. $$\frac{1}{1+u^2} = \frac{1}{2}\left(\frac{1}{1-iu} + \frac{1}{1+iu}\right) = \frac{1}{2}\left(\int_{-\infty}^{0} e^{x-iux} dx + \int_0^{\infty} e^{-x-iux} dx\right).$$

7. This definition is more practical than the one in [16] where $\mathcal{C}E$ played the rôle of E.

8. If $p=0$ then (11) disappears from the statement of the lemma so that (12) implies (13) under the given hypotheses.

9. Indeed, if for $u_1 < u_2$ we found

$$\min\left(-u_1 F'/F, u_1^2\right) > \min\left(-u_2 F'/F, u_2^2\right) = u_2^2,$$

   say, this would mean that the min $(-u_1 F'/F, u_1^2)$ was incorrectly chosen since $u_1^2 < u_2^2$.

BIBLIOGRAPHY

1. AHLFORS, "Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen," *Acta Societatis Scientiarum Fennicae*, (N.S.) A, I (1930), no. 9.
2. AHIEZER and BABENKO, "On weighted polynomials of approximation to functions continuous on the whole real axis," *Doklady Akad. Nauk S.S.S.R.* (N.S.) 57 (1947).
3. BANACH, "Théorie des opérations linéaires," Warsaw (1932).
4. CARLEMAN, "L'intégrale de Fourier et questions qui s'y rattachent," Uppsala (1944).
5. FERRAND, "Extension d'une inégalité de M. Ahlfors," *Comptes Rendus*, Paris, t. 220 (1945).

5a. DUFRESNOY, "Sur un théorème d'Ahlfors et son application à l'étude de la représentation conforme," *ibid.*

6. FERRAND, "Note on a paper by Mandelbrojt and MacLane," *Trans. Am. Math. Soc.*, Vol. 61 (1947).
7. HILLE, "Functional analysis and semi-groups," *Am. Math. Soc. Coll. Publ.*, Vol. 31 (1948).
8. HORVÀTH, "Sur l'approximation polynomiale des fonctions sur une demi-droite," *Comptes Rendus*, Paris, t. 227 (1948).
9. MANDELBROJT, "Analytic functions and classes of infinitely differentiable functions," *The Rice Institute Pamphlet*, Vol. XXIX (1942), p. 1–142.
10. MANDELBROJT, "Quasi-analyticity and properties of flatness of entire functions," *Duke Math. Jour.*, Vol. 9 (1942).
11. MANDELBROJT, "Sur une inégalité fondamentale," *Ann. École Norm. Sup.*, (3), t. 63 (1946).
12. MANDELBROJT and G. MACLANE, "On functions holomorphic in a strip region, and an extension of Watson's problem," *Trans. Am. Math. Soc.*, Vol. 61 (1947).
13. MANDELBROJT, "Théorèmes d'unicité," *Ann. École Norm. Sup.*, (3), t. 65 (1948).
14. MANDELBROJT and AGMON, "Une généralisation du théorème taubérien de Wiener," *Acta Scientiarum Mathematicarum*, Szeged, Vol. 12, Part B (1950).
15. MANDELBROJT, "Un théorème de fermeture," *Comptes Rendus*, Paris, t. 231 (1950).
16. MANDELBROJT, "Théorèmes généraux de fermeture," *ibid.*, t. 232 (1951).
17. H. J. REITER, "On functions holomorphic in a strip and Mandelbrojt's inequality," M.A. Thesis, The Rice Institute, Houston, 1950.
18. L. SCHWARTZ, "Théorie générale des fonctions moyenne-périodiques" *Annals of Math* (1947).
19. SHOHAT and TAMARKIN, "The problem of moments," *Am. Math. Soc., Mathematical Surveys*, No. 1 (1943).
20. WARSCHAWSKI, "On conformal mapping of infinite strips," *Trans. Am. Math. Soc.*, Vol. 51 (1942).
21. WIENER, "The Fourier integral and certain of its applications," Cambridge (1933).